Mathematical Methods, Part 1: Applied Intertemporal Optimization

Part II

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Part II
Stochastic Models in Discrete Time

3 Setting up a random environment

- In a stochastic world, all quantities take the form of random variables
- We will first review some basic concepts from probability theory required for our purposes
- Following structure is very condensed. You are strongly encouraged to consult Stachurski (2009) for a more in-depth treatment.
- In the sequel we work with infinite discrete time periods $\mathbb{T} = \{0, 1, 2, \ldots\}$.
- If A is any set, 2^A or $\mathrm{Pow}(A)$ denotes the power set, i.e., the class of all subsets of A.

3.1 Basic concepts from probability theory

3.1.1 Probability spaces and random variables

- Randomness in our model enters via an exogenous stochastic process $(\theta_t)_{t\geq 0}$, i.e., a sequence of random variables with values in $\Theta \subset \mathbb{R}^N$, $N \geq 1$.

- All these random variables live on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where:

    - Ω is the *sample space* which represents all possible states of the world
    - $\mathscr{F} \subset \mathrm{Pow}(\Omega)$ is a collection of subsets of Ω that form a σ*-algebra*, i.e., (i) $\Omega \in \mathscr{F}$, (ii) $A \in \mathscr{F}$ implies $A^c := \Omega \backslash A \in \mathscr{F}$ and (iii) $(A_n)_{n\geq 0}$, $A_n \in \mathscr{F}\ \forall n$ implies $\cup_{n=0}^{\infty} A_n \in \mathscr{F}$.
    - $\mathbb{P} : \mathscr{F} \longrightarrow [0,1]$ is a probability measure, i.e., a countably additive function satisfying $\mathbb{P}(\Omega) = 1$ that assigns probabilities $\mathbb{P}(A)$ to each measurable subset $A \in \mathscr{F}$ of Ω

- Θ is endowed with some σ-algebra $\mathscr{A} \subset \mathrm{Pow}(\Theta)$ to become a measurable space $(\Theta, \mathscr{A})$

- Since $\Theta \subset \mathbb{R}^N$ is a topological space, we can (and typically do) choose for $\mathscr{A}$ the Borel-σ algebra $\mathscr{B}(\Theta)$ which is the smallest σ-algebra containing the topology

- For each $t \in \mathbb{T}$, the mapping $\theta_t : \Omega \longrightarrow \Theta$ is $\mathscr{F} - \mathscr{B}(\Theta)$ measurable, i.e., for all $B \in \mathscr{B}(\Theta)$, $\theta_t^{-1}(B) := \{\omega \in \Omega | \theta_t(\omega) \in B\} \subset \Omega$ is an element of $\mathscr{F}$.

3.1.2 Probability and distributions of random variables

- For each $t \in \mathbb{T}$, can construct probability distribution/measure μ_t of random variable θ_t:
  - given a set $B \in \mathscr{B}(\Theta)$, $\mu_t(B)$ is the probability that $\theta_t \in B$
  - straightforward to construct μ_t by defining the *image measure*

$$\mu_t(B) = \mathbb{P}\left(\theta_t^{-1}(B)\right) = \mathbb{P}\left(\{\omega \in \Omega | \theta_t(\omega) \in B\}\right) \tag{81}$$

  - mapping $\mu_t : \mathscr{B}(\Theta) \longrightarrow [0,1]$ is indeed a probability measure on $(\Theta, \mathscr{B}(\Theta))$ and called the probability distribution of θ_t
  - if $\Theta = \mathbb{R}$, there is a one-to one correspondence between distribution μ_t and the *distribution function* $F_t(b) := \mu_t(]-\infty, b])$, $b \in \mathbb{R}$. Similar result holds if $N > 1$.
- Analogously, construct joint distribution $\mu_{\mathbb{I}}$ of random variables $\theta_{\mathbb{I}} := (\theta_t)_{t \in \mathbb{I}}$ for any $\mathbb{I} \subset \mathbb{T}$
- Further, can infer the distributions of random variables defined by measurable functions
  - $f : \Theta \longrightarrow \mathbb{X} \subset \mathbb{R}^M$ of θ_t
  - $f : \Theta^{\mathbb{I}} \longrightarrow \mathbb{X} \subset \mathbb{R}^M$ of $\theta_{\mathbb{I}}$

  with values in the measurable space $(\mathbb{X}, \mathscr{B}(\mathbb{X}))$

3.2 Constructing the underlying probability space

- Previous result: Given $(\Omega, \mathscr{F}, \mathbb{P})$ and measurable mappings $(\theta_t)_t$, can compute probability distributions of all random variables $(\theta_t)_{t\in\mathbb{I}}$, $\mathbb{I} \subset \mathbb{T}$ and of all measurable functions of these random variables
- Can also reverse the previous construction:
  - specify distributions/dependence structure of the random variables $(\theta_t)_{t\in\mathbb{T}}$
  - construct an underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ consistent with this.

3.2.1 Example 1: Independent random variables

- Suppose we want $(\theta_t)_{t\geq 0}$ to consist of independent random variables wit values in Θ each having a desired probability distribution $\mu : \mathscr{B}(\Theta) \longrightarrow [0,1]$, say, a normal distribution.
- In this case, define:
  - $\Omega = \Theta^{\mathbb{T}}$ (the set of sequences with values in Θ)
  - $\mathscr{F} = \mathscr{B}(\Omega)$ (the product σ-algebra generated by measurable rectangles or, equivalently, the Borel σ-algebra when Ω is endowed with the product topology)
  - $\mathbb{P} = \mu^{\mathbb{T}}$ (the product measure which satisfies $\mu^{\mathbb{T}}(\Omega \times \ldots \times \Omega \times A \times B \times \Omega \times \ldots) = \mu(A) \cdot \mu(B)$ for any $A, B \in \mathscr{B}(\Theta)$)

3.2.2 Example 2: Correlated random variables

- Suppose we want $(\theta_t)_{t\in\mathbb{T}}$ to follow an auto-regressive structure of the form

$$\theta_t = M\theta_{t-1} + \varepsilon_t, \quad t \geq 1, \tag{82}$$

  where $M \in \mathbb{R}^{N\times N}$ and $(\varepsilon_t)_{t\geq 1}$ consists of i.i.d. random variables with values in $\mathcal{E} \subset \mathbb{R}^N$ and distribution μ_ε which are independent of θ_0 which has distribution μ_0.

- In this case, can also construct $(\Omega, \mathscr{F}, \mathbb{P})$ by defining $\Omega = \Theta\times\mathcal{E}^{\mathbb{N}}$, $\mathscr{F} = \mathscr{B}(\Omega)$, $\mathbb{P} = \mu_0\otimes\mu_\varepsilon^{\mathbb{N}}$.

- Noting that $\theta_t = A^t\theta_0 + \sum_{n=0}^{t-1} M^n\varepsilon_{t-n}$ we can compute μ_t for each $t > 0$ via (81)

- For later reference, note that (121) defines a *transition probability*, i.e., a mapping Q : $\Theta \times \mathscr{B}(\Theta) \longrightarrow [0,1]$ such that $Q(\theta, A)$ is the probability that $\theta_{t+1} \in A$ given that $\theta_t = \theta$

- For all $\theta \in \Theta$ and $A \in \mathscr{B}(\Theta)$, Q can explicitly be constructed as

$$Q(\theta, A) = \mu_\varepsilon\{\varepsilon \in \mathcal{E} | M\theta + \varepsilon \in A\} \tag{83}$$

- The distributions $(\mu_t)_{t\in\mathbb{T}}$ can then be computed recursively for $t \geq 1$ as

$$\mu_t(B) = \int_\Theta Q(\theta, B)\mu_{t-1}(d\theta). \tag{84}$$

  for each $B \in \mathscr{B}(\Theta)$.

3.3 Filtration and conditional expectation

- Let $(\theta_t)_{t\geq 0}$ be the exogenous stochastic process on $(\Omega, \mathscr{F}, \mathbb{P})$ defined previously.
- In our equilibrium framework derived below, all endogenous variables will take the form of random variables $(X_t)_{t\geq 0}$ with values in $\mathbb{X} \subset \mathbb{R}^M$ which depend on the exogenous process.
- We generally take the notation $(X_t)_{t\geq 0}$ to mean that X_t is observable in period t, i.e., can only depend on exogenous random variables θ_n, $n \leq t$.
- To impose this restriction formally, define a *filtration* $(\mathscr{F}_t)_{t\geq 0}$ where $\mathscr{F}_t \subset \mathscr{F}$ is the smallest σ-algebra such that each θ_n, $0 \leq n \leq t$ is $\mathscr{F}_t$-$\mathscr{B}(\Theta)$ measurable.
- Process $(X_t)_{t\in\mathbb{T}}$ is said to be *adapted* (to $(\mathscr{F}_t)_{t\geq 0}$) if each X_t is $\mathscr{F}_t - \mathscr{B}(\mathbb{X})$ measurable. This captures exactly the idea that X_t can depend only on random variables θ_n, $n \leq t$
- Specifically, if $(X_t)_{t\in\mathbb{T}}$ is adapted and $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathscr{F}_t]$ is the expectations operator conditional on observations up to time t, $\mathbb{E}_t[X_n] = X_n$ for all t and $n \leq t$.
- If X_t has distribution $\mu_{X_t} : \mathscr{B}(\mathbb{X}) \longrightarrow [0,1]$ and is integrable, its unconditional expectation is defined as

$$\mathbb{E}[X_t] := \int_\Omega X_t(\omega)\mathbb{P}(d\omega) = \int_\mathbb{X} x\mu_{X_t}(dx). \tag{85}$$

4 Stochastic decision problems with finite horizon

4.1 A stochastic OLG model

- Consider a stochastic version of the OLG model from Section 1.6 similar to Wang (1993):
  - all assumptions on population structure, labor supply, etc. remain the same
  - but: production side modified to incorporate random production shocks
- We continue to denote equilibrium variables as $(X_t)_{t\geq 0}$ but these are now adapted stochastic processes rather than just sequences.
- All equalities and inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely without explicit notice.

4.1.1 Production side

- Suppose that production in period t is subject to multiplicative shock $\theta_t \in \Theta \subset \mathbb{R}_{++}$:

$$Y_t = \theta_t F(K_t, L_t) = \theta_t L_t f(k_t) \tag{86}$$

- Production shocks $(\theta_t)_{t\geq 0}$ consists of independent random variables with distribution μ and values in $\Theta = [\theta_{\min}, \theta_{\max}] \subset \mathbb{R}_{++}$.

- Thus, we can chose the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ exactly as outlined in Section 3.2.1

- In each period t, the firm takes the current shock in period t as given and decides on demand for capital and labor.

- Continue to impose Assumption 1.3 on f and define $k_t = \frac{K_t}{L_t}$ for all $t \in \mathbb{T}$ as before.

- The fist order conditions then determine equilibrium factor prices as:

$$w_t = \mathcal{W}(k_t, \theta_t) := \theta_t[f(k_t) - k_t f'(k_t)] \tag{87a}$$
$$R_t = \mathcal{R}(k_t, \theta_t) := \theta_t f'(k_t) \tag{87b}$$

4.1.2 A stochastic two-period decision problem

- Consider decision problem of a young consumer in period $t \geq 0$:
  - consumer observes her current labor income $w_t > 0$ (which is a real number)
  - capital return R_{t+1} treated as random variable with values in $[R_{\min}, R_{\max}] \subset \mathbb{R}_{++}$
  - knowing the underlying probabilistic structure, consumer computes correct conditional expectation $\mathbb{E}_t[\cdot]$ of next period's random variables
- Any investment decision $s_t \in [0, w_t]$ (which is a number!) determines lifetime consumption

$$c_t^y = w_t - s_t \tag{88a}$$

$$c_{t+1}^o = R_{t+1} s_t \tag{88b}$$

  where $c_t^y \geq 0$ is a number and c_{t+1}^o a random variable with values in $[s_t R_{\min}, s_t R_{\max}]$.

- Preferences over alternative random variables (c_t^y, c_{t+1}^o)possess an expected utility representation with von-Nemann Morgenstern utility $U(c^y, c^o) = u(c^y) + \beta u(c^o)$
- Decision problem reads:

$$\max_s \left\{ u(w_t - s) + \beta \mathbb{E}_t \left[u(s R_{t+1}) \right] \mid 0 \leq s \leq w_t \right\} \tag{89}$$

- Define consumer's objective function $U_t :]0, w_t[\longrightarrow \mathbb{R}$,

$$U_t(s) := u(w_t - s) + \beta \mathbb{E}_t \left[u(sR_{t+1}) \right] \tag{90}$$

- Imposing Assumption 1.2 on utility u, we obtain the following result:

Lemma 4.1 *Under Assumption 1.2, the following holds:*

(i) U_t *in (90) is* C^2*, strictly concave, and* $\lim_{s \searrow 0} U_t'(s) = -\lim_{s \nearrow w_t} U_t'(s) = -\infty$

(ii) Problem (89) has a unique interior solution s_t^* *determined by*

$$u'(w_t - s) = \beta \mathbb{E}_t \left[R_{t+1} u'(sR_{t+1}) \right] \tag{91}$$

- **Hint:** When proving this result, exploit that in the present case, differentiation can be interchanged with the expectations operator

4.1.3 Deriving the equilibrium equations

- Aggregate investment made at time t determines next periods's capital stock $K_{t+1} = Ns_t$
- Defining $k_{t+1} = K_{t+1}/N$, (91) can be written as:

$$u'(w_t - k_{t+1}) = \beta \mathbb{E}_t \left[R_{t+1} u'(k_{t+1} R_{t+1}) \right] \tag{92}$$

- Observations:
  - by (87b), next period's capital return determined by $R_{t+1} = \theta_{t+1} f'(k_{t+1})$
  - uncertainty in R_{t+1} completely due uncertainty about shock which has distribution μ independent of any other realizations at time t
  - this permits (92) to be written as:

$$\begin{aligned} u'(w_t - k_{t+1}) &= \beta \mathbb{E}_\mu \left[\mathcal{R}(k_{t+1}, \cdot) u'(k_{t+1} \mathcal{R}(k_{t+1}, \cdot)) \right] && (93) \\ &= \beta \int_\Theta \mathcal{R}(k_{t+1}, \theta) u'(k_{t+1} \mathcal{R}(k_{t+1}, \theta)) \mu(d\theta). \end{aligned}$$

- Consumption of both generations in t satisfies:

$$c_t^y = w_t - k_{t+1} \tag{94a}$$

$$c_t^o = R_t k_t. \tag{94b}$$

4.1.4 Equilibrium

- Stochastic OLG economy is summarized by the list $\mathscr{E}_{SOLG} = \langle u, \beta, f, \mu \rangle$
- Following definition of equilibrium is straightforward generalization of deterministic case.

Definition 4.1 *Given $k_0 > 0$, an equilibrium of $\mathscr{E}_{SOLG}$ consists of adapted stochastic processes of prices $(w_t^e, R_t^e)_{t \geq 0}$ and an allocation $(k_{t+1}^e, c_t^{y,e}, c_t^{o,e})_{t \geq 0}$ satisfying (87), (93), and (94) for all $t \geq 0$.*

- Questions as in the deterministic case:
  - existence of equilibrium?
  - uniqueness of equilibrium?
  - dynamic behavior of equilibrium?
- To answer them, will again derive recursive structure of equilibrium.

4.1.5 Recursive structure of equilibrium

- Following ideas exactly analogous to deterministic case studied in Section 1.6.5
- Given $k > 0$ and $\theta \in \Theta$, define for each $0 < k_+ < \mathcal{W}(k,\theta)$ the function

$$H(k_+; k, \theta) := u'(\mathcal{W}(k,\theta) - k_+) - \beta \mathbb{E}_\mu \left[\mathcal{R}(k_+, \cdot)\, u'(k_+ \mathcal{R}(k_+, \cdot)) \right]. \qquad (95)$$

- Equilibrium process $(k^e_{t+1})_{t \geq 0}$ solves $H(k_{t+1}; k_t, \theta_t) = 0$ for all $t \geq 0$ and determines all other equilibrium variables
- Following auxiliary result can be proved exactly as in the deterministic case:

Lemma 4.2 *Under Assumptions 1.2 and 1.3, the following holds:*

(i) The function $H(\cdot; k, \theta)$ defined in (95) has at least one zero for all $k > 0$ and $\theta \in \Theta$.

(ii) If, in addition either (a) or (b) of Assumption hold, this zero is unique.

- Lemma 4.2 allows us to state the following main result:

  Proposition 4.1 *Under Assumptions 1.2 and 1.3, the following holds for all $k_0 > 0$:*

  (i) Economy $\mathscr{E}_{\text{SOLG}}$ has at least one equilibrium.

  (ii) If, in addition, either (a) or (b) of Assumption 1.4 hold, this equilibrium is unique.

- Observations:

  - additional restrictions ensure existence of a map $\mathcal{K} : \mathbb{R}_{++} \times \Theta \longrightarrow \mathbb{R}_{++}$ which determines the unique solution $k_+ = \mathcal{K}(k, \theta)$ to (34) for each $k > 0$ and $\theta \in \Theta$
  - by the implicit function theorem, $\mathcal{K}$ is C^1, strictly increasing, and satisfies

$$0 < \mathcal{K}(k, \theta) < \mathcal{W}(k, \theta) < f(k, \theta). \tag{96}$$

  - unique equilibrium process $(k^e_{t+1})_{t \geq 0}$ determined recursively by

$$k^e_{t+1} = \mathcal{K}(k^e_t, \theta_t). \tag{97}$$

- To study equilibrium dynamics, need some basic concepts from stochastic dynamical systems theory

5 Stochastic dynamical systems in discrete time

5.1 Stochastic dynamical systems and stability

- For more details, the reader is again referred to Stachurski (2009).
- Assume that endogenous state dynamics take the form $F : \mathbb{X} \times \Theta \longrightarrow \mathbb{X}$

  $$x_{t+1} = F(x_t, \theta_t) \tag{98}$$

  where we now restrict attention to case where $\mathbb{X} = \mathbb{R}_+$
- Also assume that exogenous process is i.i.d. with distribution μ_θ and values in $\Theta = [\theta_{\min}, \theta_{\max}] \subset \mathbb{R}_{++}$
- In the deterministic case, the state x_t in period t is a real number
- In the stochastic case, the state x_t in period t is a random variable which is completely described by its distribution $\mu_t : \mathscr{B}(\mathbb{X}) \longrightarrow [0, 1]$
- Thus, a steady state in the stochastic case is a distribution $\bar{\mu}$ (or a random variable $\bar{x}$ which has this distribution) which remains invariant under (98).
- Thus, to compute a stochastic steady state of (98), we need to derive how the sequence of distributions $(\mu_t)_{t\geq 0}$ of the random variables $(x_t)_{t\geq 0}$ evolve over time

5.2 Markov operator

- Suppose x_0 has distribution μ_0, what is the distribution μ_t of x_t for any $t \geq 1$?
- As in Section 3.2.2, note that (98) defines a *transition probability*, i.e., a mapping $Q : \mathbb{X} \times \mathscr{B}(\mathbb{X}) \longrightarrow [0, 1]$ such that $Q(x, A)$ is the probability that $x_{t+1} \in A$ given that $x_t = x$
- For all $x \in \mathbb{X}$ and $A \in \mathscr{B}(\mathbb{X})$, Q can explicitly be constructed as

$$Q(x, A) = \mu_\theta\{\theta \in \Theta | F(x, \theta) \in A\} \tag{99}$$

- The distributions $(\mu_t)_{t\in\mathbb{T}}$ can then be computed recursively for $t \geq 1$ as

$$\mu_t(B) = \int_{\mathbb{X}} Q(x, B)\mu_{t-1}(dx). \tag{100}$$

  for each $B \in \mathscr{B}(\mathbb{X})$.

- Let $\mathscr{M}(\mathbb{X})$ denote the class of probability measures on $\mathscr{B}(\mathbb{X})$
- Then, can define an operator $T : \mathscr{M}(\mathbb{X}) \longrightarrow \mathscr{M}(\mathbb{X})$ which associates with any $\mu \in \mathscr{M}(\mathbb{X})$ the new probability measure $T\mu$ defined for each $B \in \mathscr{B}(\mathbb{X})$ as

$$T\mu(B) = \int_{\mathbb{X}} Q(x, B)\mu(dx). \tag{101}$$

5.3 Stochastic steady states

- The concept of an invariant distribution is now straightforward:

  Definition 5.1 *An steady state of the stochastic dynamical system (98) is a probability distribution* $\bar{\mu} \in \mathscr{M}(\mathbb{X})$ *which is a fixed point of* T*, i.e.,* $T\bar{\mu} = \bar{\mu}$*.*

- The stochastic analog of a steady state is therefore an invariant probability distribution
- Large literature which studies existence of invariant distributions for Markov operators
- Notion of stability requires $\lim_{t\to\infty} T^t \mu_0 = \bar{\mu}$ where the limiting operation requires a suitable notion of convergence of measures (most results on stability use the concept of weak convergence, see Stokey & Lucas (1989)).
- Very general conditions for existence/uniqueness/stablity of invariant distributions if F resp. T has certain monotonicity properties in Kamihigashi & Stachurski (2014)
- There is also a theory of *Random Dynamical Systems* due to Arnold (1998) which defines the concept of a random fixed point.
- See Schenk-Hoppé & Schmalfuss (2001) for an economic application of this theory and how it relates to the previous concepts

5.4 Equilibrium dynamics in the stochastic OLG model

5.4.1 Stable sets

- Consider existence of stochastic steady states/invariant distributions of economy $\mathscr{E}_{\text{SOLG}}$
- Existence of stochastic steady states follows from the existence of *stable sets*:

Definition 5.2 *A stable set of (97) is an interval* $[k_{\min}, k_{\max}] \subset \mathbb{R}_{++}$ *such that:*

(i) $\mathcal{K}(k_{\min}, \theta_{\min}) = k_{\min}$

(ii) $\mathcal{K}(k_{\max}, \theta_{\max}) = k_{\max}$

(iii) $\mathcal{K}(k, \theta_{\min}) < k < \mathcal{K}(k, \theta_{\max})$ *for all* $k \in [k_{\min}, k_{\max}]$

- Existence of a stable set non-trivial steady state $\bar{k} > 0$ not guaranteed, fails if

$$\mathcal{K}(k, \theta_{\max}) < k \tag{102}$$

  for all $k > 0$ (impoverishment).

- Sufficient condition to exclude this and ensure existence is

$$\lim_{k \searrow 0} \mathcal{K}'(k, \theta_{\min}) > 1. \tag{103}$$

5.4.2 Existence of a stochastic steady state

- Following existence result due to Wang (1993):

  Proposition 5.1 (Wang (1993)) *If the equilibrium map $\mathcal{K}$ from (97) satisfies condition (103), there exists at least one stochastic steady state/invariant probability distribution.*

- Uniqueness of a stable sets not guaranteed, same multiplicity problem as in the deterministic case.

- Uniqueness obtains, however, if for all $\theta \in \Theta$, $\mathcal{K}(\cdot, \theta)$ has a unique fixed point.

- This is a special case of the more general concept of a *stable fixed point configuration* (cf. Brock & Mirman (1972)). Essentially, this requires that the largest fixed point of $\mathcal{K}(\cdot, \theta_{\min})$ be smaller than the smallest fixed point of $\mathcal{K}(\cdot, \theta_{\max})$ (cf. the illustrations provided in class).

- Much more general existence results on stochastic steady states that also hold for a much larger class of OLG economies can be found, e.g., in Morand & Reffett (2007) and McGovern, Morand & Reffett (2013).

6 Stochastic decision problems with infinite horizon

6.1 A prototype decision problem

- Consider the problem as in Section 2.2 with $\mathbb{T} := \{0, 1, 2, \ldots\}$ but now with uncertainty
- In particular, we now:
  - abstract from loans by requiring $s_t \geq 0$.
  - include the consumer's labor-leisure choice $h_t \in [0, \bar{h}]$ which determines labor supply
- Remainder normalizes maximum labor to $\bar{h} = 1$.

6.1.1 Decision setup

- Given variables:
  - adapted stochastic process of wages $w^\infty = (w_t)_{t\in\mathbb{T}}$
  - adapted stochastic process of capital returns $R^\infty = (R_t)_{t\in\mathbb{T}}$
  - initial capital $\bar{s}_{-1} \geq 0$
- Decision variables:
  - consumption plan: adapted stochastic process $(c_t)_{t\in\mathbb{T}} \geq 0$, $c_t \geq 0\ \forall t$
  - investment plan: adapted stochastic process $(s_t)_{t\in\mathbb{T}}$, $s_t \geq 0\ \forall t$
  - labor supply plan: adapted stochastic process $(h_t)_{t\in\mathbb{T}}$, $0 \leq h_t \leq \bar{h}\ \forall t$

6.1.2 Intertemporal budget set

- Feasible plans must satisfy period budget equation

$$c_t + s_t \leq w_t h_t + R_t s_{t-1} \tag{104}$$

for all $t \in \mathbb{T}$ where $s_{-1} = \bar{s}_{-1}$

- Feasible plans are defined by budget set:

$$\mathbb{B}(w^\infty, R^\infty, \bar{s}_{-1}) = \left\{(c_t, h_t, s_t)_{t\in\mathbb{T}} | \, c_t \geq 0, 0 \leq h_t \leq 1, s_t \geq 0, (104) \text{ holds for all } t \in \mathbb{T}\right\} \tag{105}$$

6.1.3 Preferences and decision problem

- Utility in period t now depends on consumption $c_t \geq 0$ and leisure $0 \leq h_t \leq 1$ and given by utility function

$$u : \mathbb{R}_+ \times [0,] \longrightarrow \mathbb{R}, \ (c,h) \mapsto u(c,h) \tag{106}$$

- Preferences over consumption-labor processes $(c_t, h_t)_{t\in\mathbb{T}}$ represented by utility function

$$U((c_t,h_t)_{t\in\mathbb{T}}) := \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t u(c_t,h_t)\right], \quad 0<\beta<1. \tag{107}$$

- Decision problem:

$$\max\left\{U((c_t,h_t)_{t\in\mathbb{T}})\middle|(c_t,h_t,s_t)_{t\in\mathbb{T}} \in \mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1})\right\}. \tag{108}$$

Assumption 6.1 *The utility function u in (106) is continuous, strictly concave and C^2 on the interior of its domain with partial derivatives satisfying*

$$\partial_{cc}u < 0 < \partial_c u \qquad \textit{and} \qquad \lim_{c\searrow 0}\partial_c u(c,h) = \infty \tag{109a}$$

$$\partial_{hh}u < 0 < -\partial_h u \qquad \textit{and} \qquad \lim_{h\nearrow 1}\partial_h u(c,h) = -\infty. \tag{109b}$$

6.2 Solving the decision problem

- Following derivations impose Assumption 6.1
- Then, any solution to (108) will be interior, i.e., $c_t^* > 0$ and $0 < h_t^* < 1$ due to (109)
- Can again use a variational argument to obtain following equations which characterize solution
- For each $t \geq 0$ and conditional on $\mathscr{F}_t$, solution to (108) must satisfy the intratemporal optimality condition

$$-\frac{\partial_h u(c_t, h_t)}{\partial_c u(c_t, h_t)} = w_t \tag{110}$$

  and the intertemporal optimality condition (Euler equation)

$$\mathbb{E}_t \left[R_{t+1} \frac{\beta \partial_c u(c_{t+1}, h_{t+1})}{\partial_c u(c_t, h_t)} \right] = 1. \tag{111}$$

- Further, for all $t \geq 0$, the budget equality

$$c_t + s_t = w_t h_t + R_t s_{t-1} \tag{112}$$

holds and the stochastic transversality condition (STVC)

$$\lim_{T\to\infty} \mathbb{E}_0 \left[s_T \prod_{t=0}^{T} R_t^{-1} \right] = 0. \tag{113}$$

- Remarks:
  - that any process $(c_t^*, s_t^*, h_t^*)_{t\in\mathbb{T}} \in \mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1})$ satisfying (110), (111), (112) for all $t \in \mathbb{T}$ as well as (113) is indeed a solution to (108) can be proved along the lines of the proof of Proposition 2.3 done in class (exploiting the law of iterated expectations!).
  - one can also show by using the same arguments as in the proof of Proposition 2.3 that the solution to (108) is $\mathbb{P}$-a.s. unique.
  - we could also - somewhat mechanically - have used a Lagrangian approach to obtain the previous conditions, but it is not quite clear how derivatives conditional on $\mathscr{F}_t$ should be interpreted.

6.3 An equilibrium framework: The RBC model

- Consider a stochastic version of the neoclassical growth model from Section 2.4 with endogenous labor supply:
  - production side modified to incorporate random production shocks
  - consumer side modified to include labor-leisure choice, decision problem solved under uncertainty as in Section 6.1
  - unless stated otherwise, all other assumptions remain the same as in Section 2.4
- We continue to denote equilibrium variables as $(X_t)_{t\geq 0}$ but these are now adapted stochastic processes rather than just sequences.
- All equalities and inequalities involving random variables are assumed to hold $\mathbb{P}$-almost surely without explicit notice.

6.3.1 Consumer side

- As in deterministic case, N identical consumers who each
  - plan over infinitely many future periods $\mathbb{T} = \{0, 1, 2, \ldots\}$
  - consume c_t and invest s_t in period t
  - supply h_t units of labor in period t, now determined endogenously
  - capital earns return R_t, labor the wage w_t in t
- Decision problem exactly as in Section 6.1
- As consumers are identical, so are the decisions they take!
- At the aggregate level, factor supply in period $t \geq 0$ given by

$$L_t = Nh_t \tag{114}$$

$$K_t = Ns_{t-1} \tag{115}$$

and capital per capita $k_t := K_t/N$ evolves as

$$k_t = s_{t-1}, \quad t \geq 1. \tag{116}$$

- Optimal decision satisfies the conditions:

$$\begin{aligned} c_t + k_{t+1} &= w_t h_t + R_t k_t && (117a) \\ -\frac{\partial_h u(c_t, h_t)}{\partial_c u(c_t, h_t)} &= w_t && (117b) \\ \mathbb{E}_t \left[R_{t+1} \frac{\beta \partial_c u(c_{t+1}, h_{t+1})}{\partial_c u(c_t, h_t)} \right] &= 1 && (117c) \end{aligned}$$

and the stochastic transversality condition (STVC)

$$\lim_{T \to \infty} \mathbb{E}_0 \left[k_{T+1} \prod_{t=0}^{T} R_t^{-1} \right] = 0. \quad (118)$$

6.3.2 Production side

- Suppose that (net) production in period t is subject to multiplicative shock $\theta_t \in \Theta \subset \mathbb{R}_{++}$ such that total output (including non-depreciated capital) is given by:

$$Y_t = e^{\theta_t} F(K_t, L_t) + (1-\delta)K_t = N\left[e^{\theta_t} h_t f(k_t/h_t) + (1-\delta)k_t\right] \tag{119}$$

- Continue to assume linear homogeneity of F and impose Assumption 1.3 on f.

- Using (114) and (115) and linear homogeneity, per capita output $y_t := Y_t/N$ given by

$$y_t = e^{\theta_t} F(k_t, h_t) + (1-\delta)k_t = e^{\theta_t} h_t f(k_t/h_t) + (1-\delta)k_t \tag{120}$$

- Production shocks $(\theta_t)_{t\geq 0}$ follow an AR(1)-process of the form

$$\theta_t = \rho\theta_{t-1} + \varepsilon_t \tag{121}$$

  where $0 \leq \rho < 1$ and $(\varepsilon_t)_{t\geq 0}$ consist of i.i.d. random variables with distribution μ_ε

- Thus, we can chose the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and construct transition probability Q induced by (121) exactly as outlined in Section 3.2.2

- In each period t, the firm takes the current shock in period t as given and decides on demand for capital and labor.

- The fist order conditions then determine equilibrium factor prices as:

$$\begin{aligned} w_t = \mathcal{W}(k_t, h_t, \theta_t) &:= e^{\theta_t} \partial_h F(k_t, h_t) \qquad (122a) \\ &= e^{\theta_t} \left[f(k_t/h_t) - k_t/h_t f'(k_t/h_t) \right] \\ R_t = \mathcal{R}(k_t, h_t, \theta_t) &:= e^{\theta_t} \partial_k F(k_t, h_t) + (1 - \delta) \qquad (122b) \\ &= e^{\theta_t} f'(k_t/h_t) + 1 - \delta \end{aligned}$$

- Remark:
  - in the deterministic case, non-depreciated capital was included in F resp. f which had the interpretation of a gross production function
  - here, we interpret F resp. f as a net production function and must, therefore, explicitly keep track of non-depreciated capital
  - the reason is that only net production output is affected by the shock.

6.4 Equilibrium

- Economy is summarized by the list $\mathscr{E}_{RBC} = \langle u, \beta, N, f, Q \rangle$ plus initial conditions $k_0 > 0$ and $\theta_0 \in \Theta$

Definition 6.1 *Given $k_0 > 0$ and $\theta_0 \in \Theta$, an equilibrium of $\mathscr{E}_{RBC}$ is an allocation $(c_t^e, h_t^e, k_{t+1}^e)_{t\geq 0}$ and a price sequence $(w_t^e, R_t^e)_{t\geq 0}$ which satisfy (117) and (122) for all $t \geq 0$ and (118).*

- Can again use an equivalent planning problem to determine the (unique) equilibrium allocation

- Equilibrium prices then follow directly from (122) for all $t \geq 0$.

6.5 A stochastic planning problem

- Consider a benevolent social planner who maximizes consumer utility by choosing a feasible allocation.

  Definition 6.2 *Given $k_0 > 0$ and $\theta_0 \in \Theta$, a feasible allocation is an adapted stochastic process $(c_t, h_t, k_{t+1})_{t \geq 0}$ which satisfies $c_t \geq 0$, $0 \leq h_t \leq 1$, $k_{t+1} \geq 0$ for all $t \geq 0$ as well as the resource constraint*

$$k_{t+1} + c_t \leq e^{\theta_t} F(k_t, h_t) + (1 - \delta) k_t. \tag{123}$$

  The set of feasible allocations is denoted $\mathbb{A}(k_0, \theta_0)$.

- The planning problem reads:

$$\max_{(c_t, h_t, k_{t+1})_{t \in \mathbb{T}}} \left\{ U((c_t, h_t)_{t \in \mathbb{T}}) \,\middle|\, (c_t, h_t, k_{t+1})_{t \in \mathbb{T}} \in \mathbb{A}(k_0, \theta_0) \right\} \tag{124}$$

- As in the deterministic case, can compute the equations that characterize the solution to (124)
- Can show that these coincide with the equilibrium equations derived above.
- Thus, the solution to (68) also constitutes an equilibrium allocation!

6.6 Solving the stochastic planning problem by recursive methods

6.6.1 The Bellman equation

- Motivation for the following approach is analogous to the deterministic case
- Basic idea: Exploit the recursive structure of SPP
- Assume that f satisfies Assumption 1.3 and u Assumption 6.1 and $0 < \beta < 1$
- For brevity, set

$$M(k, h, \theta) := e^{\theta} F(k, h) + (1 - \delta)k \tag{125}$$

- In the present stochastic setup, the Bellmann equation reads:

$$V(k, \theta) = \max_{k_+ \geq 0, 0 \leq h \leq 1} \left\{ u(M(k, h, \theta) - k_+, h) + \beta \int_{\Theta} V(k_+, \theta_+) Q(\theta, d\theta_+) \,|\, k_+ \leq M(k, h, \theta) \right\} \tag{126}$$

6.6.2 Policy function

- Having computed the value function V, suppose the maximizing solution (k_+^*, h^*) in (126) is well-defined and unique for each $(k, \theta) \in \mathbb{R}_{++} \times \Theta$
- Define the policy function $g = (g_k, g_h) : \mathbb{R}_{++} \times \Theta \longrightarrow \mathbb{R}_+ \times [0, 1]$

$$g(k, \theta) = \arg \max_{k_+ \geq 0, 0 \leq h \leq 1} \left\{ u(M(k, h, \theta) - k_+, h) + \beta \int_\Theta V(k_+, \theta_+) Q(\theta, d\theta_+) \,|\, k_+ \leq M(k, h, \theta) \right\}$$

Lemma 6.1 *Let V be the unique solution to (126) and $g = (g_k, g_h)$ be defined as above. Then, for each (k_0, z_0) the sequence $\{c_t^\star, h_t^\star, k_{t+1}^\star\}_{t \geq 0}$ defined recursively as $k_0^\star = k_0$,*

$$\begin{aligned} k_{t+1}^\star &= g_k(k_t^\star, \theta_t) \\ h_t^\star &= g_h(k_t^\star, \theta_t) \\ c_t^\star &= M(k_t^\star, h_t^\star, \theta_t) - k_{t+1}^\star \end{aligned}$$

for all $t \geq 0$ is a solution to (124).

6.7 Equilibrium dynamics in the RBC model

- Consequences of previous results:
  - dynamics completely described by the endogenous state variable $\{k_t^\star\}_{t\geq 0}$ and the exogenous process $\{\theta_t\}_{t\geq 0}$
  - analogously to stochastic OLG model, can analyze dynamics, existence of invariant distributions, etc.
  - in general, mapping $g_k(\cdot;\theta)$ has a unique steady state $\bar{k}_\theta$ for all $\theta \in \Theta$

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