Solution to Problem Set 1

Problem 1.1.
(a) As $0 < \sigma < 1$, u satisfies Assumptions 1.1 and 1.2. The decision problem is written as

$$\max_{\{c_t, s_t\}_{t\in\mathbb{T}}} \left\{ \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \right.$$

subject to

$$\left. \begin{aligned} c_t + s_t &\leq e_t + R_t s_{t-1}, && (1)\\ c_t &\geq 0, && (2)\\ s_{-1} &\text{ given}, && (3)\\ \text{and } s_T &= 0 \text{ for all } t \in \mathbb{T} && (4) \end{aligned} \right\}$$

(b) We can solve the problem using Kuhn-Tucker Lagrangian with inequality constraints, but life would be much easier if we could argue that (1) is binding and (2) is non-binding at the optimum. Suppose (1) is not binding at the optimum, one can increase c_t such that (1) is binding and the objective function is strictly increased. It is straightforward that zero consumption does not maximize utility. (2) must hold with strict inequality and can therefore be dropped. The problem can be rewritten as follows.

$$\max_{\{c_t, s_t\}_{t\in\mathbb{T}}} \left\{ \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} \right.$$

subject to

$$\left. \begin{aligned} c_t + s_t &= e_t + R_t s_{t-1}, && (5)\\ s_{-1} &\text{ given}, && (6)\\ \text{and } s_T &= 0 \text{ for all } t \in \mathbb{T} && (7) \end{aligned} \right\}$$

(i) Lagrangian approach

$$\mathcal{L}((c_t, s_t, \lambda_t)_{t\in\mathbb{T}}) = \sum_{t=0}^{T} \beta^t \Big[\frac{c_t^{1-\sigma} - 1}{1-\sigma} - \lambda_t (c_t + s_t - e_t - R_t s_{t-1}) \Big] \tag{8}$$

First order conditions

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t} = 0 : &\quad c_t^{-\sigma} = \lambda_t \\ \frac{\partial \mathcal{L}}{\partial s_t} = 0 : &\quad \lambda_t = \beta R_{t+1} \lambda_{t+1} \\ \frac{\partial \mathcal{L}}{\partial \lambda_t} = 0 : &\quad c_t + s_t = e_t + R_t s_{t-1} \end{aligned}$$

and (6) and (7) for all $t \in \mathbb{T}$. Eliminating the Lagrangian multiplier we get

$$c_{t+1} = (\beta R_{t+1})^{1/\sigma} c_t = \beta^{1/\sigma} \Big(\frac{q_t}{q_{t+1}} \Big)^{1/\sigma} c_t \tag{9}$$

$$s_t = e_t + R_t s_{t-1} - c_t \tag{10}$$

and (6) and (7) for all $t \in \mathbb{T}$, where $q_t := (R_1 \cdot R_2 \cdots R_t)^{-1}$.

Using the budget constraint to eliminate s_t we obtain

$$\sum_{t=1}^{T} q_t c_t = \sum_{t=0}^{T} q_t e_t + R_0 s_{-1} =: M \tag{11}$$

Using (9) to rewrite (11)

$$c_0 \sum_{t=0}^{T} (\beta^t q_t^{\sigma-1})^{1/\sigma} = M \tag{12}$$

This results in

$$c_0 = \bar{c}_0 M \quad \text{where} \quad \bar{c}_0 := \left(\sum_{t=0}^{T} (\beta^t q_t^{\sigma-1})^{1/\sigma} \right)^{-1}. \tag{13}$$

So for any period t, we obtain

$$c_t = \bar{c}_t M / q_t \quad \text{where} \quad \bar{c}_t := \Big(\beta^t q_t^{\sigma-1} \Big)^{1/\sigma} \Big/ \left(\sum_{t=0}^{T} (\beta^t q_t^{\sigma-1})^{1/\sigma} \right). \tag{14}$$

The solution to s_t is then determined using (10)

(ii) Recursive method

With the argument above, the problem in a recursive form is given as

$$V_t(W) = \max_{c,s} \Big\{ u(c) + \beta V_{t+1}(e_{t+1} + s R_{t+1}) | c + s = W, s \geq -E_t \Big\} \tag{15}$$

By substitution, the value function can be rewritten as

$$V_t(W) = \max_{s} \Big\{ u(W - s) + \beta V_{t+1}(e_{t+1} + s R_{t+1}) | s \geq -E_t \Big\} \tag{16}$$

Combing the first order condition and the envelope condition and using the specific utility function should result in the optimality conditions

$$c_{t+1} = (\beta R_{t+1})^{1/\sigma} c_t := \beta^{1/\sigma} \Big(\frac{q_t}{q_{t+1}} \Big)^{1/\sigma} c_t \tag{17}$$

$$s_t = e_t + R_t s_{t-1} - c_t \tag{18}$$

And the terminal conditions continue to hold. Here we can apply the same algorithm as in (i) to obtain the same solution to c_t and s_t. One more thing you can do is compute the value function for each period t by guessing its functional form. This is a bit involved and left as an exercise.

(c) First notice that

$$\sum_{t=0}^{T} \bar{c}_t = 1 \tag{19}$$

Therefore, (14) implies that the optimal consumption expenditure $q_t c_t$ each period is a fraction $\bar{c}_t$ of discounted lifetime income M. The consumption share $\bar{c}_t$ is exclusively determined by consumption prices $(q_t)_{t \in \mathbb{T}}$ and independent of these prices if $\sigma = 1$.

(d) $\sigma < 1$ ensures that u is well-defined for $c = 0$ as required by Assumption 1.1/1.2. When $\sigma \geq 1$, the problem can be slightly modified as discussed in Section 1.5 in class.

$$\max_{\{c_t, s_t\}_{t \in \mathbb{T}}} \Big\{ \Big\} \sum_{t=0}^{T} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$c_t + s_t \leq e_t + R_t s_{t-1},$$

$$c_t \geq \underline{c},\ \underline{c} > 0 \text{ small enough},$$

$$s_{-1} \text{ given},$$

$$\text{and } s_T = 0 \text{ for all } t \in \mathbb{T} \Big\}$$

Choosing $\underline{c} > 0$ sufficiently small ensures that the condition $c_t \geq \underline{c}$ is non-binding for all t and the associated Lagrangian multiplier is zero. One can then proceed as above to obtain the unique solution which retains the same functional form as with $\sigma < 1$. Thus, all previous results are valid for all $\sigma > 0$.

Problem 1.2.
(a) Young consumer's decision problem

$$\max_{k_{t+1}} \Big\{ \log(w_t - k_{t+1}) + \beta \log(k_{t+1} R_{t+1}) | 0 \leq k_{t+1} \leq w_t \Big\} \tag{20}$$

The optimality conditions are written as

$$\frac{\beta c_t^y}{c_{t+1}^o} = \frac{1}{R_{t+1}} \tag{21}$$

$$c_t^y = w_t - k_{t+1} \tag{22}$$

$$c_{t+1}^o = R_{t+1} k_{t+1} \tag{23}$$

(b) The firm's decision problem

$$\max_{K_t, L_t} \Big\{ A K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - R_t K_t | (K_t, L_t) \in \mathbb{R}_+^2 \Big\} \tag{24}$$

The first order conditions are

$$R_t = \alpha A K_t^{\alpha-1} L_t^{1-\alpha} = \alpha A k_t^{\alpha-1} \tag{25}$$

$$w_t = (1-\alpha) A K_t^{\alpha} L_t^{-\alpha} = (1-\alpha) A k_t^{\alpha} \tag{26}$$

where $k_t = K_t / L_t$.
(c) Equations (21)-(23), (25)-(26) characterize the general equilibrium. By substitution, one can obtain

$$k_{t+1} = \frac{\beta}{1+\beta} (1-\alpha) A k_t^{\alpha} \tag{27}$$

(d) So the function $\mathcal{K}(k_t)$ is given as $\mathcal{K}(k_t) = \frac{\beta}{1+\beta}(1-\alpha) A k_t^{\alpha}$, which is strictly increasing and strictly concave (you should be able to sketch it).