Marten Hillebrand, Hoang Van Khieu **Mathematical Methods, Part 1**

Winter 2015/16

Problem Set 2

Problem 2.1. Consider the following dynamical systems in discrete time:

(i) $\mathbb{X} = \mathbb{R}_+$, $\varphi = \mathcal{K}$ with $\mathcal{K}$ defined as part (d) of Problem 1.2.

(ii) $\mathbb{X} = [-1, 1]$, $\varphi(x) = Bx^3 + (1 - B)x$, $0 < B < 4$

Analyze each case by proceeding as follows:

(a) Verify that $(\varphi, \mathbb{X})$ is indeed a dynamical system, i.e., φ maps $\mathbb{X}$ into itself.

(b) Compute all fixed points of φ and analyze their local stability properties (depending on the parameter B in case (ii)).

(c) Illustrate the global dynamic behavior in a diagram. (Hint: Distinguish the cases $0 < B < 1$, $1 < B < 2$, and $B > 2$ in case (ii). You may therefore plot φ for $B = \frac{1}{2}$, $B = \frac{3}{2}$, and $B = 4$).

Problem 2.2. Consider the class of linear difference equations on $\mathbb{X} = \mathbb{R}^2$ generated by time one-map

$$\varphi(x) = Ax \tag{1}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}. \tag{2}$$

Recall that the Eigenvalues of A are the solutions λ_1, λ_2 to the condition $\chi_A(\lambda) := \det[A - \lambda I_2] = 0$ where I_2 is the 2×2 identity matrix. Denote by $\mathrm{tr}(A) := a_{11} + a_{22}$ the trace and $\det(A)$ the determinant of A.

(i) Derive a condition on A under which φ has a unique steady state $\bar{x}$.

(ii) Derive a formula which determines the Eigenvalues λ_1, λ_2 of A as a function of $\mathrm{tr}(A)$ and $\det(A)$. Which restrictions ensure that both Eigenvalues are real/complex?

(iii) Use your previous formula to determine all fixed points $\bar{x}$ and their stability properties for the following cases:

(a) $A_1 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{2} \end{bmatrix}$

(b) $A_2 = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{6} & 0 \end{bmatrix}$

(c) $A_3 = \begin{bmatrix} 1 & \frac{1}{4} \\ 9 & 1 \end{bmatrix}$

(iv) In case (c) in (iii), determine the so-called *stable manifold*, i.e., the subset $\mathbb{M} \subset \mathbb{X}$ of initial values $x_0 \in \mathbb{X}$ with the property that $\lim_{n\to\infty} \varphi^n(x_0) = \bar{x}$. Hint: Compute $\mathbb{M}$ as the Eigenspace associated with the smaller Eigenvalue. Sketch the set $\mathbb{M}$ in a diagram.

Enjoy!

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