# Solution to Problem Set 3

**Problem 3.1**.
**(a)** First we assume that $0 < \sigma < 1$, for the case $\sigma \geq 1$, we can apply the same argument as in Problem 1.1.d. Define the budget set

$$\mathbb{B}(e^\infty, R^\infty, \bar{s}_{-1}) = \{(c_t, s_t)_{t\in\mathbb{T}} | c_t \geq 0, c_t + s_t \leq e_t + R_t s_{t-1} \text{ for all } t \in \mathbb{T}, s_{-1} = \bar{s}_{-1} \text{ given }, \lim_{t\to\infty} q_t s_t \geq 0\},$$

where $\mathbb{T}$ is defined as in class. The lifetime utility function is written as

$$U((c_t)_{t\in\mathbb{T}}) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

Iterating the budget constraint and using the NPG condition, one can obtain

$$\sum_{t=0}^{\infty} q_t c_t \leq \sum_{t=0}^{\infty} q_t e_t + R_0 \bar{s}_{-1} := M,$$

where we impose $0 < M < \infty$, and $q_t := (R_1 \cdot R_2 \cdots R_t)^{-1}$ as in class. Now define the lifetime budget set

$$\mathcal{B}(M, q, s_{-1}) := \Big\{(c_t)_{t\in\mathbb{T}} | \sum_{t=0}^{\infty} q_t c_t \leq M \wedge c_t \geq 0 \quad \forall t \in \mathbb{T} \wedge \ s_{-1} = \bar{s}_{-1} \text{ given}\Big\}$$

where $q := (q_t)_{t\in\mathbb{T}}$.

So, the decision problem is written as

$$\max_{\{c_t\}_{t\in\mathbb{T}}} \Big\{U((c_t)_{t\in\mathbb{T}}) | (c_t)_{t\in\mathbb{T}} \in \mathcal{B}(M)\Big\}$$

**(b)** By the same argument in Problem 1.1.b, the Lagrangian is written as

$$\mathcal{L}((c_t)_{t\in\mathbb{T}}, \lambda) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma} - \lambda\Big(\sum_{t=0}^{\infty} q_t c_t - M\Big) \tag{1}$$

First order conditions

$$\frac{\partial \mathcal{L}}{\partial c_t} = 0 : \quad c_t^{-\sigma} = \lambda q_t \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 : \quad \sum_{t=0}^{\infty} q_t c_t = M \tag{3}$$

for all $t \in \mathbb{T}$, and $s_{-1} = \bar{s}_{-1}$ given and $\lim_{t\to\infty} q_t s_t^* = 0$ (TVC) must hold. As (2) holds for all $t$, (2) can be rewritten as

$$c_{t+1} = \beta^{1/\sigma}\Big(\frac{q_t}{q_{t+1}}\Big)^{1/\sigma} c_t = (\beta R_{t+1})^{1/\sigma} c_t \tag{4}$$

Using (4) to rewrite (3)

$$c_0 \sum_{t=0}^{\infty} (\beta^t q_t^{\sigma-1})^{1/\sigma} = M \tag{5}$$

This results in

$$c_0 = \bar{c}_0 M \quad \text{where} \quad \bar{c}_0 := \left( \sum_{t=0}^{\infty} (\beta^t q_t^{\sigma-1})^{1/\sigma} \right)^{-1}. \tag{6}$$

So for any period $t$, we obtain

$$c_t = \bar{c}_t M / q_t \quad \text{where} \quad \bar{c}_t := \left(\beta^t q_t^{\sigma-1}\right)^{1/\sigma} \Big/ \left( \sum_{t=0}^{\infty} (\beta^t q_t^{\sigma-1})^{1/\sigma} \right). \tag{7}$$

The solution to $s_t$ is then determined using the period budget constraint.

**(c)** We easily see that

$$\sum_{t=0}^{\infty} \bar{c}_t = 1 \tag{8}$$

Therefore, (7) implies that the optimal consumption expenditure $q_t c_t$ each period is a fraction $\bar{c}_t$ of discounted lifetime income $M$. The consumption share $\bar{c}_t$ is exclusively determined by consumption prices $(q_t)_{t \in \mathbb{T}}$ and independent of these prices if $\sigma = 1$.

**Problem 1.2**.
**(i)** Given arbitrary $W_1 = e_1 + R_1 s_0 \geq -E_1$, the decision problem is written as

$$V_1(W_1) = \max_{c_1, s_1} \Big\{ \log(c_1) + \beta \log(e_2 + s_1 R_2) | c_1 \geq 0, c_1 + s_1 \leq W_1, s_1 \geq -E_1 \Big\}$$

**(ii)** We can argue that $c > 0$ and that the budget constraint must bind at the optimum. So, the problem can be rewritten as

$$V_1(W_1) = \max_{s_1} \Big\{ \log(W_1 - s_1) + \beta \log(e_2 + s_1 R_2) | s_1 \geq -E_1 \Big\}$$

The first order condition is written as

$$S_1(W_1) = \frac{\beta W_1 - e_2/R_2}{1+\beta} \tag{9}$$

$C_1(W_1)$ is determined by $W_1 - S_1(W_1)$. Thus

$$C_1(W_1) = W_1 - \frac{\beta W_1 - e_2/R_2}{1+\beta} = \frac{W_1 + e_2/R_2}{1+\beta} \tag{10}$$

**(iii)**

$$\begin{aligned} V_1(W_1) &= \log(W_1 - S_1(W_1)) + \beta \log(e_2 + S_1(W_1) R_2) \\ &= \log\Big(\frac{W_1 + e_2/R_2}{1+\beta}\Big) + \beta \log\Big(\frac{\beta R_2 (W_1 + e_2/R_2)}{1+\beta}\Big) \\ &= (1+\beta) u\Big(\frac{W_1 + e_2/R_2}{1+\beta}\Big) + \beta \log(\beta R_2) \end{aligned}$$

Now we take derivative of $V_1$ w.r.t $W_1$ and obtain

$$V_1'(W_1) = \frac{1+\beta}{W_1 + e_2/R_2} \tag{11}$$

We also have

$$u'(W_1 - S_1(W_1)) = u'(C_1(W_1)) = \frac{1+\beta}{W_1 + e_2/R_2} \tag{12}$$

So, we conclude $V_1'(W_1) = u'(W_1 - S_1(W_1)) = u'(C_1(W_1))$.
**(iv)**

$$V_0(W_0) = \max_{s_0}\Big\{\log(W_0 - s_0) + \beta V_1(W_1) | s_0 \geq -E_0\Big\} \tag{13}$$

**(v)** FOC:

$$\frac{1}{c_0} = \beta R_1 V_1'(W_1) \tag{14}$$

Using (11), we obtain

$$\begin{aligned}
c_0^* &= \frac{W_1^* + e_2/R_2}{R_1\beta(1+\beta)} &(15)\\
&= \frac{e_1/R_1 + s_0^* + e_2/(R_1R_2)}{\beta(1+\beta)} &(16)\\
&= \frac{e_1/R_1 + (e_0 + R_0 s_{-1} - c_0^*) + e_2/(R_1R_2)}{\beta(1+\beta)} &(17)\\
&= \frac{e_0 + e_1 q_1 + e_2 q_2 + R_0 s_{-1}}{1+\beta+\beta^2} &(18)
\end{aligned}$$

This solution $c_0^*$ is the same as in Problem 1.1 when $\sigma = 1$ and $T = 2$. Solution $s_0^*$ can be derived as below.

$$s_0^* = e_0 + R_0 s_{-1} - c_0^* \tag{19}$$

**(vi)**

$$\begin{aligned}
c_1^* &= \frac{W_1^* + e_2/R_2}{1+\beta} &(20)\\
&= \frac{e_1 + R_1 s_0^* + e_2/R_2}{1+\beta} &(21)\\
&= \frac{e_1 + R_1(e_0 + R_0 s_{-1} - c_0^*) + e_2/R_2}{1+\beta} &(22)\\
&= \frac{e_0 + e_1 q_1 + e_2 q_2 + R_0 s_{-1} - c_0^*}{q_1(1+\beta)} &(23)\\
&= \frac{\beta}{1+\beta+\beta^2}\frac{e_0 + e_1 q_1 + e_2 q_2 + R_0 s_{-1}}{q_1} &(24)
\end{aligned}$$

$s_1^*$ is determined by $s_1^* = e_1 + R_1 s_0^* - c_1^*$. So, we can see that the solutions $c_1^*$ and $s_1^*$ are the same as in Problem 1.1 when $\sigma = 1$ and $T = 2$.