Solution to Problem Set 4

Problem 4.1.
(a) The Bellman equation written as

$$V(k) = \max_{c,k_+}\{\log(c) + \beta V(k_+)|c = Ak^\alpha - k_+ \wedge 0 \leq k_+ \leq Ak^\alpha\} \tag{1}$$

(b) Now we guess $V(k) = a + b\log(k)$. As V is concave, we must have $0 < k_+ < Ak^\alpha$ at the optimum. So, the Bellman equation can be rewritten as

$$a + b\log(k) = \max_{k_+}\{\log(Ak^\alpha - k_+) + \beta(a + b\log(k_+)\} \tag{2}$$

FOC:

$$\frac{1}{Ak^\alpha - k_+} = \frac{\beta b}{k_+} \tag{3}$$

This implies

$$k_+ = \frac{\beta Abk^\alpha}{1 + \beta b} \tag{4}$$

As (2) is maximized at k_+ determined by (4), plugging (4) into (2) gives

$$a + b\log(k) = \log\Big(\frac{Ak^\alpha}{1+\beta b}\Big) + \beta(a + b\log\Big(\frac{\beta bAk^\alpha}{1+\beta b}\Big) \tag{5}$$

$$= \alpha(1 + \beta b)\log(k) + (1 + \beta b)\log\Big(\frac{A}{1+\beta b}\Big) + \beta a + \beta b\log(\beta b) \tag{6}$$

(6) implies that

$$b = \alpha(1 + \beta b) \tag{7}$$

$$a = (1 + \beta b)\log\Big(\frac{A}{1+\beta b}\Big) + \beta a + \beta b\log(\beta b) \tag{8}$$

$$\Rightarrow b = \frac{\alpha}{1 - \alpha\beta} \tag{9}$$

We can ignore a because it is not of interest. Given b in (9), the policy function can be rewritten as

$$k_+ = \alpha\beta Ak^\alpha \tag{10}$$

(c) The non-trivial fixed point of the policy function $\bar{k}$ is determined by

$$\bar{k} = \alpha\beta A\bar{k}^\alpha \tag{11}$$

$$\Rightarrow \bar{k} = (\alpha\beta A)^{1/(1-\alpha)} \tag{12}$$

Using the same argument in Problem 2.1.i.b, we conclude that $\bar{k}1$ is locally stable.
(d) The policy functions in this problem and in Problem 1.2 are different as the saving rates in the OLG and neo-classical growth models are are different. The saving rate the the neo-classical growth model is higher if $\alpha > \frac{1}{2+\beta}$. You should be able to plot the policy function.

Problem 4.2.
(a) The Bellman equation written as

$$V(k, c_{-1}) = \max_{c,k_+}\{\log(c) + \gamma\log(c_{-1}) + \beta V(k_+, c)|c = Ak^\alpha - k_+ \wedge 0 \leq k_+ \leq Ak^\alpha\} \tag{13}$$

(b) Now we guess $V(k) = E + F\log(k) + G\log(c_{-1})$. As V is concave, we must have $0 < k_+ < Ak^\alpha$ at the optimum. So, the Bellman equation can be rewritten as

$$\begin{aligned}&E + F\log(k) + G\log(Ak_{-1}^\alpha - k) = \\ &\max_{k_+}\{\log(Ak^\alpha - k_+) + \gamma\log(Ak_{-1}^\alpha - k) + \beta(E + F\log(k_+) + G\log(Ak^\alpha - k_+))\}\end{aligned} \tag{14}$$

FOC:

$$\frac{1 + \beta G}{Ak^\alpha - k_+} = \frac{\beta F}{k_+} \tag{15}$$

This implies

$$k_+ = \frac{\beta F A k^\alpha}{1 + \beta(F + G)} \tag{16}$$

As (14) is maximized at k_+ determined by (16), plugging (16) into (14) gives

$$\begin{aligned}&E + F\log(k) + G\log(c_{-1}) \\ &= \log\Big(\frac{Ak^\alpha(1 + \beta G)}{1 + \beta(F + G)}\Big) + \gamma\log(Ak_{-1}^\alpha - k) + \beta(E + F\log\Big(\frac{\beta F A k^\alpha}{1 + \beta(F + G)}\Big) + G\log\Big(\frac{Ak^\alpha(1 + \beta G)}{1 + \beta(F + G)}\Big) \\ &= \alpha(1 + \beta G + \beta F)\log(k) + \gamma\log(c_{-1}) + (1 + \beta F + \beta G)(\log(A) - \log(1 + \beta(F + G))) \\ &+ (1 + \beta G)\log(1 + \beta G) + \beta E + \beta F\log(\beta F)\end{aligned} \tag{17}$$

(17) implies that

$$E = (1 + \beta F + \beta G)(\log(A) - \log(1 + \beta(F + G))) + (1 + \beta G)\log(1 + \beta G) + \beta E + \beta F\log(\beta F) \tag{18}$$
$$F = \alpha(1 + \beta G + \beta F) \tag{19}$$
$$G = \gamma \tag{20}$$

(19) and (20) yields

$$F = \frac{\alpha(1 + \beta\gamma)}{1 - \alpha\beta} \tag{21}$$

We can ignore E because it is not of interest. So, the policy function can be rewritten as

$$k_+ = \alpha\beta A k^\alpha \tag{22}$$

(c) So the policy functions in cases with and without *habit persistence* are the same, meaning the the saving rates in the two cases are equal. Intuitively, with habit persistence in utility, today's consumption affects tomorrow's utility through γ. Also, tomorrow's consumption affects the following period's utility, and so on. It turns out that these effects are cancelled out, resulting in the same saving rate as without habit persistence. To sum up, past consumption is irrelevant for forward-looking individuals in this case.