

Solution to Problem Set 6

Problem 6.1.

(a) Consumer's problem reads

$$\max_{\{c_t, h_t, k_{t+1}\}_{t=0}^{\infty}} \left\{ \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t (\log(c_t) + v(1 - h_t)) \right] \middle| c_t + k_{t+1} \leq w_t h_t + R_t k_t \text{ for all } t \right. \\ \left. \wedge c_t \geq 0 \wedge 0 \leq h_t \leq 1 \wedge k_{t+1} \geq 0 \wedge k_0 > 0 \text{ given} \right\}$$

Optimality conditions:

$$1 = \beta \mathbb{E}_t \left[R_{t+1} \frac{c_t}{c_{t+1}} \right] \quad (1)$$

$$w_t = v'(1 - h_t) c_t \quad (2)$$

$$c_t + k_{t+1} = w_t h_t + R_t k_t \quad (3)$$

(b) The firm's decision problem reads

$$\max_{k_t, h_t} \left\{ k_t^\alpha (A_t h_t)^{1-\alpha} - w_t h_t - R_t k_t \mid (k_t, h_t) \in \mathbb{R}_+^2 \right\}$$

The first order conditions are

$$R_t = \alpha k_t^{\alpha-1} (A_t h_t)^{1-\alpha} = \alpha F(k_t, A_t h_t) / k_t \quad (4)$$

$$w_t = (1 - \alpha) k_t^\alpha A_t^{1-\alpha} h_t^{-\alpha} = (1 - \alpha) F(k_t, A_t h_t) / h_t \quad (5)$$

(4) and (5) imply that $w_t h_t + R_t k_t = F(k_t, A_t h_t)$. Therefore,

$$c_t = F(k_t, A_t h_t) - k_{t+1} = (1 - s_t) F(k_t, A_t h_t).$$

(c) Using (4) to rewrite (1) we obtain

$$1 = \alpha \beta \mathbb{E}_t \left[\frac{F(k_{t+1}, A_{t+1} h_{t+1})}{c_{t+1}} \frac{c_t}{k_{t+1}} \right] \\ = \alpha \beta \mathbb{E}_t \left[\frac{F(k_{t+1}, A_{t+1} h_{t+1})}{(1 - s_{t+1}) F(k_{t+1}, A_{t+1} h_{t+1})} \frac{(1 - s_t) F(k_t, A_t h_t)}{s_t F(k_t, A_t h_t)} \right] \\ = \alpha \beta \mathbb{E}_t \left[\frac{1 - s_{t+1}}{1 - s_t} \frac{1}{s_t} \right]$$

Assume the saving rate is constant, then it follows that

$$1 = \alpha \beta \mathbb{E}_t \left[\frac{1 - s}{1 - s} \frac{1}{s} \right],$$

which results in $s = \alpha \beta$.

(d) Using (5) to rewrite (2) we have

$$v'(1 - h_t) h_t = \frac{1 - \alpha}{1 - \alpha \beta} \quad (6)$$

Define $v'(1 - h_t)h_t =: g(h_t)$. Consider the first derivative of g

$$g'(h_t) = v'(1 - h_t) - h_tv''(1 - h_t)$$

As v is strictly increasing and strictly concave, and $h_t \geq 0$, it follows that $g'(h_t) > 0$. Thus, g is invertible and

$$h_t = g^{-1}\left(\frac{1 - \alpha}{1 - \alpha\beta}\right) =: \bar{h},$$

which is constant.

(e)

$$k_{t+1} = sF(k_t, A_t h_t) = \alpha\beta k_t^\alpha (A_t h_t)^{1-\alpha} = \alpha\beta \bar{h}^{1-\alpha} A_t^{1-\alpha} k_t^\alpha \equiv \mathcal{K}(k_t, A_t) \quad (7)$$

The lower and upper bound capital of the stable set is determined by

$$\begin{aligned} k_{\min} &= \alpha\beta \bar{h}^{1-\alpha} \underline{A}^{1-\alpha} k_{\min}^\alpha = \bar{h} \underline{A} (\alpha\beta)^{1/1-\alpha} \\ k_{\max} &= \alpha\beta \bar{h}^{1-\alpha} \bar{A}^{1-\alpha} k_{\max}^\alpha = \bar{h} \bar{A} (\alpha\beta)^{1/1-\alpha} \end{aligned}$$

Consider the limit

$$\lim_{k \searrow 0} \mathcal{K}'(k, \underline{A}) = \lim_{k \searrow 0} \alpha^2 \beta \bar{h}^{1-\alpha} \underline{A}^{1-\alpha} k^{\alpha-1} = \infty > 1$$

So, the stable set is

$$[k_{\min}, k_{\max}] = [\bar{h} \underline{A} (\alpha\beta)^{1/1-\alpha}, \bar{h} \bar{A} (\alpha\beta)^{1/1-\alpha}]$$

$$c_t = (1 - s)F(k_t, A_t h_t) = (1 - \alpha\beta) k_t^\alpha (A_t h_t)^{1-\alpha} = (1 - \alpha\beta) \bar{h}^{1-\alpha} A_t^{1-\alpha} k_t^\alpha \equiv \mathcal{C}(k_t, A_t) \quad (8)$$

The lower and upper bound consumption of the stable set is determined by

$$\begin{aligned} c_{\min} &= \mathcal{C}(k_{\min}, \underline{A}) =: (1 - \alpha\beta) \bar{h}^{1-\alpha} \underline{A}^{1-\alpha} k_{\min}^\alpha = \bar{h} \underline{A} (1 - \alpha\beta) (\alpha\beta)^{\alpha/1-\alpha} \\ c_{\max} &= \mathcal{C}(k_{\max}, \bar{A}) =: (1 - \alpha\beta) \bar{h}^{1-\alpha} \bar{A}^{1-\alpha} k_{\max}^\alpha = \bar{h} \bar{A} (1 - \alpha\beta) (\alpha\beta)^{\alpha/1-\alpha} \end{aligned}$$

Consider the limit

$$\lim_{k \searrow 0} \mathcal{C}'(k, \underline{A}) = \lim_{k \searrow 0} (1 - \alpha\beta) \alpha \bar{h}^{1-\alpha} \underline{A}^{1-\alpha} k^{\alpha-1} = \infty > 1$$

So, the stable set is

$$[c_{\min}, c_{\max}] = [\bar{h} \underline{A} (1 - \alpha\beta) (\alpha\beta)^{\alpha/1-\alpha}, \bar{h} \bar{A} (1 - \alpha\beta) (\alpha\beta)^{\alpha/1-\alpha}]$$

Problem 6.2.

(a) The Bellman equation is written as

$$\begin{aligned} V(k, A) &= \max_{k_+, c, h} \{ \log(c) + \log(1 - h) + \beta \mathbb{E}[V(k_+, A_+)] \} c + k_+ = k^\alpha (Ah)^{1-\alpha} \\ &\quad \wedge c \geq 0 \wedge k_+ \geq 0 \wedge 0 \leq h \leq 1 \end{aligned} \quad (9)$$

where $wh + Rk = k^\alpha (Ah)^{1-\alpha}$, which is predetermined in the firm's problem.

(b) Now we guess $V(k, A) = B + C \log(k) + D \log(A)$. The Bellman equation can be rewritten as

$$B + C \log(k) + D \log(A) = \max_{k_+, h} \{ \log(k^\alpha (Ah)^{1-\alpha} - k_+) + \log(1 - h) + \beta \mathbb{E}[(B + C \log(k_+) + D \log(A_+))] \}$$

FOCs:

$$\frac{1}{k^\alpha (Ah)^{1-\alpha} - k_+} = \frac{\beta C}{k_+} \quad (10)$$

$$\frac{1}{1-h} = \frac{(1-\alpha)k^\alpha h^{-\alpha} A^{1-\alpha}}{k^\alpha (Ah)^{1-\alpha} - k_+} \quad (11)$$

Rearranging (10) and (11) yields

$$k_+ = \frac{\beta C k^\alpha (Ah)^{1-\alpha}}{1 + \beta C} \quad (12)$$

$$h = \frac{(1-\alpha)(1 + \beta C)}{1 + (1-\alpha)(1 + \beta C)} \equiv \bar{h} \quad (13)$$

Plugging (12) and (13) into the Bellman equation gives

$$\begin{aligned} B + C \log(k) + D \log(A) &= \alpha(1 + \beta C) \log(k) + \log(1 - \bar{h}) + \log\left(\frac{(A\bar{h})^{1-\alpha}}{1 + \beta C}\right) \\ &\quad + \beta C \log\left(\frac{\beta C (A\bar{h})^{1-\alpha}}{1 + \beta C}\right) + \beta B + \beta \mathbb{E}[\log(A_+)] \end{aligned} \quad (14)$$

(14) implies that

$$C = \alpha(1 + \beta C) = \frac{\alpha}{1 - \alpha\beta}$$

The parameters B and D are solvable but are not of interest. So the policy function for capital is given by

$$k_+ = \alpha\beta k^\alpha (Ah)^{1-\alpha}$$

Labor supply is constant and equals

$$\bar{h} = \frac{1 - \alpha}{2 - \alpha(\beta + 1)}$$

(c) Notice that when $v(1-h) = \log(1-h)$, (6) implies that $\bar{h} = \frac{1-\alpha}{2-\alpha(\beta+1)}$. The saving rate we obtained here is also constant and equals $\alpha\beta$. In conclusion, the two methods yield the same solution.