Solution to Problem Set 6

Problem 6.1.
(a) Consumer's problem reads

$$\max_{\{c_t,h_t,k_{t+1}\}_{t=0}^{\infty}} \Big\{ \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t(\log(c_t)+\upsilon(1-h_t))\Big]\Big| c_t+k_{t+1}\le w_t h_t + R_t k_t \text{ for all } t$$
$$\wedge\ c_t \ge 0\ \wedge\ 0\le h_t\le 1\ \wedge\ k_{t+1}\ge 0\ \wedge\ k_0>0 \text{ given}\Big\}$$

Let $(c_t^*, h_t^*)_{t\ge 0}$ denote the optimal consumption and labor. Consider the following perturbation

$$c_t' = c_t^* - \delta \tag{1}$$
$$c_{t+1}' = c_{t+1}^* + R_{t+1}\delta \tag{2}$$

and $c_s' = c_s^*$ for all other periods s. Thus, we have

$$\mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t(\log(c_t^*)+\upsilon(1-h_t^*))\Big] \ge \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t(\log(c_t')+\upsilon(1-h_t^*))\Big]$$
$$\iff \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t\log(c_t^*)\Big] \ge \mathbb{E}_0\Big[\sum_{t=0}^{\infty}\beta^t\log(c_t')\Big]$$
$$\iff \log(c_t^*)+\beta\mathbb{E}_t[\log(c_{t+1}^*)] \ge \log(c_t^*-\delta)+\beta\mathbb{E}_t[\log(c_{t+1}^*+R_{t+1}\delta)]$$

The right hand side is strictly concave and is maximized at $\delta=0$. Thus,

$$0 = \operatorname*{argmax}_{\delta}\{\log(c_t^*-\delta)+\beta\mathbb{E}_t[\log(c_{t+1}^*+R_{t+1}\delta)]\}$$

As δ can be either positive or negative, the following FOC is satisfied at $\delta=0$

$$1 = \beta\mathbb{E}_t\Big[R_{t+1}\frac{c_t^*}{c_{t+1}^*}\Big]$$

The perturbation for labor is left as an exercise. After all, you obtain the following optimality conditions

$$1 = \beta\mathbb{E}_t\Big[R_{t+1}\frac{c_t}{c_{t+1}}\Big] \tag{3}$$
$$w_t = \upsilon'(1-h_t)c_t \tag{4}$$
$$c_t + k_{t+1} = w_t h_t + R_t k_t \tag{5}$$

where the last condition is nothing but the budget constraint.
(b) The firm's decision problem reads

$$\max_{k_t,h_t}\Big\{k_t^{\alpha}(A_t h_t)^{1-\alpha} - w_t h_t - R_t k_t | (k_t,h_t)\in\mathbb{R}_+^2\Big\}$$

The first order conditions are

$$R_t = \alpha k_t^{\alpha-1}(A_t h_t)^{1-\alpha} = \alpha F(k_t, A_t h_t)/k_t \tag{6}$$
$$w_t = (1-\alpha)k_t^{\alpha} A_t^{1-\alpha} h_t^{-\alpha} = (1-\alpha)F(k_t, A_t h_t)/h_t \tag{7}$$

(4) and (5) imply that $w_t h_t + R_t k_t = F(k_t, A_t h_t)$. Therefore,

$$c_t = F(k_t, A_t h_t) - k_{t+1} = (1-s_t)F(k_t, A_t h_t).$$

(c) Using (4) to rewrite (1) we obtain

$$\begin{aligned} 1 &= \alpha\beta \mathbb{E}_t\Big[\frac{F(k_{t+1}, A_{t+1}h_{t+1})}{c_{t+1}} \frac{c_t}{k_{t+1}}\Big] \\ &= \alpha\beta \mathbb{E}_t\Big[\frac{F(k_{t+1}, A_{t+1}h_{t+1})}{(1-s_{t+1})F(k_{t+1}, A_{t+1}h_{t+1})} \frac{(1-s_t)F(k_t, A_t h_t)}{s_t F(k_t, A_t h_t)}\Big] \\ &= \alpha\beta \mathbb{E}_t\Big[\frac{1-s_{t+1}}{1-s_t}\frac{1}{s_t}\Big] \end{aligned}$$

Assume the saving rate is constant, then it follows that

$$1 = \alpha\beta \mathbb{E}_t\Big[\frac{1-s}{1-s}\frac{1}{s}\Big],$$

which results in $s = \alpha\beta$.
(d) Using (5) to rewrite (2) we have

$$\upsilon'(1-h_t)h_t = \frac{1-\alpha}{1-\alpha\beta} \tag{8}$$

Define $\upsilon'(1-h_t)h_t =: g(h_t)$. Consider the first derivative of g

$$g'(h_t) = \upsilon'(1-h_t) - h_t \upsilon''(1-h_t)$$

As υ is strictly increasing and strictly concave, and $h_t \geq 0$, it follows that $g'(h_t) > 0$. Thus, g is invertible and

$$h_t = g^{-1}\Big(\frac{1-\alpha}{1-\alpha\beta}\Big) =: \bar{h},$$

which is constant.
(e)

$$k_{t+1} = sF(k_t, A_t h_t) = \alpha\beta k_t^{\alpha}(A_t h_t)^{1-\alpha} = \alpha\beta \bar{h}^{1-\alpha} A_t^{1-\alpha} k_t^{\alpha} \equiv \mathcal{K}(k_t, A_t) \tag{9}$$

The lower and upper bound capital of the stable set is determined by

$$k_{\min} = \alpha\beta \bar{h}^{1-\alpha} \underline{A}^{1-\alpha} k_{\min}^{\alpha} = \bar{h}\underline{A}(\alpha\beta)^{1/1-\alpha}$$
$$k_{\max} = \alpha\beta \bar{h}^{1-\alpha} \overline{A}^{1-\alpha} k_{\max}^{\alpha} = \bar{h}\overline{A}(\alpha\beta)^{1/1-\alpha}$$

Consider the limit

$$\lim_{k \searrow 0} \mathcal{K}'(k, \underline{A}) = \lim_{k \searrow 0} \quad \alpha^2 \beta \bar{h}^{1-\alpha} \underline{A}^{1-\alpha} k^{\alpha-1} = \infty > 1$$

So, the stable set is

$$[k_{\min}, k_{\max}] = [\bar{h}\underline{A}(\alpha\beta)^{1/1-\alpha}, \bar{h}\overline{A}(\alpha\beta)^{1/1-\alpha}]$$

$$c_t = (1-s)F(k_t, A_t h_t) = (1-\alpha\beta)k_t^{\alpha}(A_t h_t)^{1-\alpha} = (1-\alpha\beta)\bar{h}^{1-\alpha} A_t^{1-\alpha} k_t^{\alpha} \equiv \mathcal{C}(k_t, A_t) \tag{10}$$

The lower and upper bound consumption of the stable set is determined by

$$c_{\min} = \mathcal{C}(k_{\min}, \underline{A}) =: (1-\alpha\beta)\bar{h}^{1-\alpha}\underline{A}^{1-\alpha}k_{\min}^{\alpha} = \bar{h}\underline{A}(1-\alpha\beta)(\alpha\beta)^{\alpha/1-\alpha}$$
$$c_{\max} = \mathcal{C}(k_{\max}, \overline{A}) =: (1-\alpha\beta)\bar{h}^{1-\alpha}\overline{A}^{1-\alpha}k_{\max}^{\alpha} = \bar{h}\overline{A}(1-\alpha\beta)(\alpha\beta)^{\alpha/1-\alpha}$$

Consider the limit

$$\lim_{k\searrow 0}\mathcal{C}'(k, \underline{A}) = \lim_{k\searrow 0} \quad (1-\alpha\beta)\alpha\bar{h}^{1-\alpha}\underline{A}^{1-\alpha}k^{\alpha-1} = \infty > 1$$

So, the stable set is

$$[c_{\min}, c_{\max}] = [\bar{h}\underline{A}(1-\alpha\beta)(\alpha\beta)^{\alpha/1-\alpha}, \bar{h}\overline{A}(1-\alpha\beta)(\alpha\beta)^{\alpha/1-\alpha}]$$

Problem 6.2.
(a) The Bellman equation is written as

$$V(k, A) = \max_{k_+,c,h}\{\log(c) + \log(1-h) + \beta\mathbb{E}[V(k_+, A_+)]|c + k_+ = k^\alpha(Ah)^{1-\alpha}$$
$$\wedge\ c \geq 0\ \wedge\ k_+ \geq 0\ \wedge\ 0 \leq h \leq 1\} \tag{11}$$

where $wh + Rk = k^\alpha(Ah)^{1-\alpha}$, which is predetermined in the firm's problem.
(b) Now we guess $V(k, A) = B + C\log(k) + D\log(A)$. The Bellman equation can be rewritten as

$$B + C\log(k) + D\log(A) = \max_{k_+,h}\{\log(k^\alpha(Ah)^{1-\alpha} - k_+) + \log(1-h) + \beta\mathbb{E}[(B + C\log(k_+) + D\log(A_+)]\}$$

FOCs:

$$\frac{1}{k^\alpha(Ah)^{1-\alpha} - k_+} = \frac{\beta C}{k_+} \tag{12}$$
$$\frac{1}{1-h} = \frac{(1-\alpha)k^\alpha h^{-\alpha}A^{1-\alpha}}{k^\alpha(Ah)^{1-\alpha} - k_+} \tag{13}$$

Rearranging (10) and (11) yields

$$k_+ = \frac{\beta C k^\alpha(Ah)^{1-\alpha}}{1+\beta C} \tag{14}$$
$$h = \frac{(1-\alpha)(1+\beta C)}{1+(1-\alpha)(1+\beta C)} \equiv \bar{h} \tag{15}$$

Plugging (12) and (13) into the Bellman equation gives

$$B + C\log(k) + D\log(A) = \alpha(1+\beta C)\log(k) + \log(1-\bar{h}) + \log\Big(\frac{(A\bar{h})^{1-\alpha}}{1+\beta C}\Big)$$
$$+\beta C\log\Big(\frac{\beta C(A\bar{h})^{1-\alpha}}{1+\beta C}\Big) + \beta B + \beta\mathbb{E}[\log(A_+)] \tag{16}$$

(14) implies that

$$C = \alpha(1+\beta C) = \frac{\alpha}{1-\alpha\beta}$$

The parameters B and D are solvable but are not of interest. So the policy function for capital is given by

$$k_+ = \alpha\beta k^\alpha(Ah)^{1-\alpha}$$

Labor supply is constant and equals

$$\bar{h} = \frac{1-\alpha}{2-\alpha(\beta+1)}$$

(c) Notice that when $v(1-h) = \log(1-h)$, (6) implies that $\bar{h} = \frac{1-\alpha}{2-\alpha(\beta+1)}$. The saving rate we obtained here is also constant and equals $\alpha\beta$. In conclusion, the two methods yield the same solution.