this "long version" is an older version of "Matching and Saving in Continuous Time: Theory". Assumption 1 here is much stronger here than in the final version. The advantage of the "long version" is that phase diagrams are derived in steps and that a phase diagram for the "intermediate interest rate case" is included (see fig. 2) which was dropped for the final version

Matching and Saving in Continuous Time: Theory

(long version)

Christian Bayer[(a)] and Klaus Wälde[(b)]

[(a)] TU Berlin and [(b)] University of Mainz, CESifo and UCL at Louvain-la-Neuve[1]

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We analyse optimal saving of risk-averse households when labour income stochastically jumps between two states. The generalized Keynes-Ramsey rule includes a precautionary savings term. A phase diagram analysis illustrates consumption and wealth dynamics between and at job transitions. There is an endogenous lower and upper limit for wealth. We derive Fokker-Planck equations for the densities of individual wealth and employment status. These equations also characterize the aggregate distribution of wealth and allow us to describe general equilibrium. We prove existence of optimal consumption paths and convergence to a unique limiting distribution in a companion paper.

JEL Codes: D91, E24, J63, J64

Keywords: matching model, optimal saving, incomplete markets, Poisson uncertainty, Fokker-Planck equations, general equilibrium

1 Introduction

Uncertain labour income is a fact of life. In the simplest conceptual representation, labour income moves stochastically between two states, high and low. Uncertainty of this type is often analyzed in continuous time setups. This is true for search and matching models à la Pissarides (1985), Burdett and Mortensen (1998) and their many applications, including e.g. the analysis of business cycles as in Shimer (2005) or mismatch as in Shimer (2007). For a recent survey of the lively search and matching literature, see Rogerson et al. (2005).

It is standard practice in this literature (for very few exceptions see below) to assume strong capital market imperfections implying that households consume their current income. Any labour market transition associated with a labour income jump

[1]Large parts of this paper were written while the authors were working at the Royal Institute of Technology (KTH) in Stockholm and the University of Glasgow, respectively. We are grateful to these institutions for their stimulating research environment. Christian Bayer: TU Berlin, Institute of Mathematics, Straße des 17. Juni 136, 10623 Berlin, bayer@math.tu-berlin.de, Klaus Wälde: University of Mainz, Mainz School of Management and Economics, Jakob-Welder-Weg 4, 55128 Mainz. klaus@waelde.com, www.waelde.com. We are very grateful to Walter Schachermayer and Josef Teichmann for comments and guidance, Michael Graber, Sevi Rodríguez Mora and Carlos Carrillo Tudela for comments and Jeremy Lise for discussions. The second author is indebted to Ken Sennewald and Christian Bauer for earlier collaboration on this topic.

therefore implies a consumption jump - consumption is far from being smooth. If households were allowed to save, however, they could smooth consumption and would generally self-insure against consumption fluctuations by accumulating wealth.

It is the objective of this paper to introduce saving into a matching framework. The first step consists in presenting and solving the maximization problem an individual faces where labour income jumps between two states. For simplicity and in the tradition of this literature, these states are called employment and unemployment even though it could also reflect periods of high and low income. The solution of this maximization problem is described by a generalized Keynes-Ramsey rule where the generalization consists in a precautionary savings term. This term lends itself to intuitive economic interpretation. For a setup with a constant interest rate and a constant labour income (as in an aggregate stationary state), the Keynes-Ramsey rule provides simple conditions under which there will be (i) consumption and wealth growth in both labour market states, (ii) growth for the employed and decline for the unemployed or (iii) decline of consumption and wealth in both labour market states.

In a second step, we provide a detailed qualitative phase-diagram analysis of the optimal behaviour of an individual, i.e. of the evolution of wealth and consumption when labour income jumps between being high and low. We can undertake a phase-diagram analysis as in continuous time deterministic setups as systems with Poisson-uncertainty are piecewise-deterministic systems: between jumps, the system evolves on continuous and differentiable trajectories. As always, a unique solution to a differential equation system requires as many boundary conditions as differential equations. We derive a boundary condition from borrowing and lending considerations which implies that the highest debt an unemployed worker can ever have is the present value of infinite unemployment benefits. This is the lower limit which is hit by an unemployed worker who dissaves. Once this limit is reached, consumption of the unemployed worker is zero and will remain zero until he finds a new job. With this boundary condition, consumption-wealth profiles are uniquely determined for both labour market states.

The third step then asks the natural question about the distribution of wealth and labour market status. Using the Dynkin formula, we obtain the Fokker-Planck equations for the wealth-employment status system. We obtain a two-dimensional partial differential equation system. It describes the evolution of the density of wealth and employment status over time, given some initial condition. When we are interested in long-run properties only, we can set time derivatives equal to zero in the Fokker-Planck equations and obtain an ordinary two-dimensional non-autonomous differential equation system. With two boundary conditions resulting from our phase diagram analysis, we obtain a unique solution and have thereby shown the existence of a stationary long-run distribution of wealth and labour market status. Convergence to this long-run distribution can be proven by building on Meyn and Tweedie (1993).

All these steps were obtained in a framework with constant interest rate, wage rate and unemployment benefit. As we want to obtain a true general equilibrium solution, we then close the model by looking at the aggregate distribution of wealth. This allows us to determine an endogenous average wealth level plus an endogenous interest and wage rate.

It is worth pointing out right at the beginning that we do not have an endogenous number of vacancies, nor do we have wage bargaining. We starting from the textbook matching model, introduced savings and removed all features which are not essential for working out the main theoretical contributions required for an introduction of savings into a matching model in continuous time. This is why we ended up working with an island-matching setup in the spirit of Lucas and Prescott (1974) where the wage is competitive.

This paper is related to various strands of the literature. First, there is a long literature that looks at the effects of labour income uncertainty which is at least partially uninsurable. In a growth model context, one can then ask - inter alia - whether the implied precautionary savings yield a higher per capita capital stock (Huggett, 1993; Aiyagari, 1994; Huggett and Ospina, 2001; Marcet et al., 2007). Second, matching and saving has been analyzed jointly in the literature starting with Andolfatto (1996) and Merz (1995). In these setups, individuals are fully insured against labour income risk as labour income is pooled in large families. Papers which exploit the advantage of CARA (constant absolute risk aversion) utility functions to jointly analyse saving and matching include Acemoglu and Shimer (1999), Hassler et al. (2005), Shimer and Werning, (2007, 2008) and Hassler and Rodriguez Mora, 1999, 2008). These papers often work with closed-form solutions for the consumption-saving decision but can not always rule out negative consumption levels for poor households. More recently, a series of general equilibrium papers (Bils et al., 2008; Nakajima, 2008; Krusell et al., 2007) does allow for individual labour income uncertainty in the presence of saving and matching and a CRRA (constant relative risk aversion) utility function. They explicitly allow for an impact of individual wealth on the outcome of the wage bargaining process and strongly rely on numerical solutions. Earlier partial equilibrium work with exogenous wages was undertaken by Lentz and Tranaes (2005) and Lentz (2009). Finally, the probably easiest way to characterize our model is to think of a Ramsey-Solow growth model with aggregate certainty where on the micro level income is uncertain due to a matching process on the labour market.

Technically, this paper builds on earlier work of one of the authors (Wälde, 1999, 2005) who analyzed optimal saving under Poisson uncertainty affecting the return to capital but not labour income.[2] We also use the insights of the long literature on optimal saving under uncertainty in continuous time. Starting with Merton (1969), it includes Turnovsky and Smith (2006), Guo et al. (2005), Bertola et al. (2005), Hassler et al. (2005), Shimer and Werning, (2007, 2008) and Hassler and Rodriguez Mora, 1999, 2008).

We view our results as complementary to the above cited discrete time approaches. The Keynes-Ramsey rules (the Euler equations) reveal a lot of economic information. The phase diagram analysis shows the wealth-employment dynamics in a very plastic way. The condition the interest rate has to satisfy such that a stationary general equi-

[2] Work completed before the present paper includes an unpublished PhD dissertation by Sennewald (2006) supervised by one of the authors which contains the derivation of the Keynes-Ramsey rules. Toche (2005) considers the saving problem of an individual where job-loss is permanent and unemployment benefits are zero. In independent work, Lise (2006) developed a Keynes-Ramsey rule for times between jumps as well.

librium consists can easily be seen. The description of distributions by differential equations allows for relatively simple existence and uniqueness proofs. These differential equations will also allow - once solved numerically - to obtain distributional information more quickly than through simulations. Finally, structural estimation as is typical for the empirical search and matching literature (van den Berg, 1990; Postel-Vinay and Robin, 2002; Flinn, 2006) can now be undertaken also for setups that include endogenous wealth accumulation.

The papers which are closest to ours are Shimer and Werning (2007, 2008) and Lise (2007). Shimer and Werning analyse unemployment insurance policies in a setup with job arrivals, deterministic or stochastic job duration and individual savings under constant absolute risk aversion. CARA preferences allow them to derive closed-form solutions which, however, can not be obtained for constant relative risk aversion as used here. Lise (2007) derives a deterministic Keynes-Ramsey rule (i.e. for periods between labour market transitions) similar to the one here for employed workers in a model with on-the-job search and firm heterogeneity. He abstracts from matching and vacancies, however, and does not provide a general equilibrium solution.

The structure of the paper is as follows. Section 2 presents the model. Section 3 derives implications of optimal behaviour including wage setting and defines general equilibrium. Section 4 presents the phase diagram analysis to understand consumption-wealth patterns over time and across labour market states. Section 5 introduces the density describing the joint distribution of the labour market status and wealth of one individual. The corresponding Fokker-Planck equations are introduced, boundary conditions are discussed and uniqueness and convergence is shown. Section 6 shows how to obtain the aggregate distribution of wealth and how to formulate appropriate initial distributions. The final section concludes. All proofs are in the companion paper by Bayer and Wälde (2009).

2 The model

We consider a model where all aggregate variables are in a steady state. At the micro level, individuals face idiosyncratic uninsurable risk.

2.1 Technologies

The production of output requires capital K and labour L. The technology is given by $Y(t) = Y(K(t), L(t))$ and $Y(.)$ has the usual neoclassical properties. In particular, it is characterized by constant returns to scale. Output can be used for consumption and for investment purposes. With a depreciation rate of δ, the aggregate capital stock follows $\dot{K}(t) = Y(t) - \delta K(t) - C(t)$.

As is common for Mortensen-Pissarides type search and matching models, the labour market is characterized by the absence of instantaneous clearing. The employment status $z(t)$ of any individual jumps between two states w and b which at the same time denote labour income in these states. As an individual cannot lose her job when she does not have one and as finding a job makes (in the absence of on-the-job search) no sense for someone who has a job, both arrival rates are state dependent.

$z(t)$	w	b
$\mu(z(t))$	0	$\mu > 0$
$s(z(t))$	$s > 0$	0

Table 1 *State dependent arrival rates*

As an example, when an individual is employed, $\mu(w) = 0$, when she is unemployed, $s(b) = 0$. The process $z(t)$ can be viewed as a continuous time Markov chain with state space $\{w, b\}$. This Markov-chain view will be used further below for the derivation of the Fokker-Planck equations describing the distributional properties of wealth and employment status.

It is most convenient for our analysis of the saving problem of the household to describe the employment process $z(t)$ by a stochastic differential equation,

$$dz(t) = \Delta(t)\, dq_\mu - \Delta(t)\, dq_s, \qquad \Delta(t) \equiv w(t) - b(t). \tag{1}$$

The Poisson process q_s counts how often our individual moves from employment into unemployment. The arrival rate of this process is given by $s(z(t))$. The Poisson process related to job finding is denoted by q_μ with an arrival rate $\mu(z(t))$. It counts how often a household leaves her "b-status", i.e. how often she finds a job.

Assume the individual is employed, $z(t) = w(t)$, then the equation for the employment status simplifies to $dz(t) = -(w(t) - b(t))\, dq_s$. Whenever the process q_s jumps, i.e. when the individual loses her job and $dq_s = 1$, the change in labour income is given by $-w(t) + b(t)$ and, given that the individual earns $w(t)$ before losing the job, earns $w(t) - w(t) + b(t) = b(t)$ afterwards. Similarly, when unemployed, the employment status follows $db = (w - b)\, dq_\mu$ and finding a job, i.e. $dq_\mu = 1$ means that labour income increases from $b(t)$ to $w(t)$.

There is a government who can tax labour income and, in an economy with market power, profits of firms. The net wage is denoted by w. Tax income from employed workers and all profits are used to finance unemployment benefits b. A static government budget constraint

$$\xi(t)\, w(t)\, L(t) = b(t)\, [N - L(t)] \tag{2}$$

is fulfilled by adjusting a proportional labour tax $\xi(t)$. The path of benefits $b(t)$ is considered to be determined by some political process which is exogenous to this model.

2.2 Preferences

We consider one individual who faces a classic consumption-saving trade-off. Labour income in the good state is given by w, in the bad state (unemployment), it amounts to $b < w$. Individuals can save in an asset a and their budget constraint reads

$$da(t) = \{ra(t) + z(t) - c(t)\}\, dt. \tag{3}$$

Per unit of time dt wealth $a(t)$ increases (or decreases) if capital income $ra(t)$ plus labour income $z(t)$ is larger (or smaller) than consumption $c(t)$. Labour income $z(t)$ is given by w when employed and b when unemployed. We will assume throughout that $b < w$. All variables with a time argument can change over time, all others, like the interest rate r and labour income w and b are constant. Dividing the budget constraint by dt and using $\dot{a}(t) \equiv da(t)/dt$ would yield a more standard expression, $\dot{a}(t) = ra(t) + z(t) - c(t)$. As $a(t)$ is not differentiable with respect to time at moments where individuals jump between employment and unemployment (or vice versa), we prefer the above representation. The latter is also more consistent with the subsequent stochastic differential equations.

The objective function of the individual is a standard intertemporal utility function,

$$U(t) = E_t \int_t^{\infty} e^{-\rho[\tau - t]} u(c(\tau))\, d\tau, \tag{4}$$

where expectations need to be formed due to the uncertainty of labour income which in turn makes consumption $c(\tau)$ uncertain. The expectations operator is E_t and conditions on the current state in t. The planning horizon starts in t (as today) and is infinite. The time preference rate ρ is positive.

Even though most of our results hold for general instantaneous utility functions, we will use the CRRA instantaneous utility function for the closed-form solution for large wealth levels (see prop. 5 below). We will then assume that

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0. \tag{5}$$

2.3 Endowment

The workforce of this economy has an exogenous and invariant size of N. The capital stock is the sum over individual wealth holdings,

$$K(t) = \Sigma_{i=1}^{N} a_i(t) = N \int ap(a,t)\, da. \tag{6}$$

The second equality, using a law of large numbers (which we will discuss more in sect. 6), expresses this sum as N times mean wealth, given a density $p(a,t)$ of wealth.

3 Optimality conditions and equilibrium

3.1 Keynes-Ramsey rules

For our understanding of optimal consumption behaviour, it is useful to derive a Keynes-Ramsey rule. We the steps suggested by Wälde (1999, 2008) for the case of an uncertain interest rate to our case of uncertain labour income. We suppress the time argument for readability but stress that these optimality rules hold for time variable factor rewards. Consumption $c(a_w, w)$ of an employed individual with current wealth a_w follows (see app. A.1)

$$-\frac{u''(c(a_w,w))}{u'(c(a_w,w))}dc(a_w,w) = \left\{r-\rho+s\left[\frac{u'(c(a_w,b))}{u'(c(a_w,w))}-1\right]\right\}dt - \frac{u''(c(a_w,w))}{u'(c(a_w,w))}\left[c(a_w,b)-c(a_w,w)\right]dq_s \tag{7}$$

while her wealth evolves according to (3) with $z = w$, i.e.

$$da_w = \left[ra_w + w - c(a_w,w)\right]dt. \tag{8}$$

Analogously, solving for the optimal consumption of an unemployed individual with current wealth a_b yields

$$-\frac{u''(c(a_b,b))}{u'(c(a_b,b))}dc(a_b,b) = \left\{r-\rho-\mu\left[1-\frac{u'(c(a_b,w))}{u'(c(a_b,b))}\right]\right\}dt - \frac{u''(c(a_b,b))}{u'(c(a_b,b))}\left[c(a_b,w)-c(a_b,b)\right]dq_\mu \tag{9}$$

and her wealth follows

$$da_b = [ra_b + b - c(a_b,b)]dt. \tag{10}$$

Without uncertainty about future labor income, i.e. $s = \mu = dq_s = dq_\mu = 0$, the above Keynes-Ramsey rules reduce to the classical deterministic consumption rule, $-\frac{u''(c)}{u'(c)}\dot{c} = (r-\rho)/\sigma$. The additional $\frac{s}{\sigma}\{.\}$ term in (7) shows that consumption growth is faster under the risk of a job loss. Similarly, the $\frac{\mu}{\sigma}\{.\}$ term in (9) shows that consumption growth for unemployed workers is smaller.

As the additional term in (7) contains the ratio of marginal utility from consumption when unemployed relative to marginal utility when employed, this suggests that it stands for precautionary savings (Leland, 1968, Aiyagari, 1994, Huggett and Ospina, 2001). When marginal utility from consumption under unemployment is much higher than marginal utility from employment, individuals experience a high drop in consumption when becoming unemployed. If relative consumption shrinks as wealth rises, i.e. if $\frac{d}{da}\frac{c(a,w)}{c(a,b)} < 0$, reducing this gap and smoothing consumption is best achieved by fast capital accumulation. This fast capital accumulation would go hand in hand with fast consumption growth as visible in (7).

In the case of unemployment, as indicated by the $\frac{\mu}{\sigma}\{.\}$ term in (9), the possibility to find a new job induces unemployed individuals to expand their momentary consumption. Relative to a situation in which unemployment is an absorbing state (once employed, always employed, i.e. $s = 0$), the prospect of a higher labor income in the future reduces the willingness to give up today's consumption. Accumulated wealth becomes less important in financing consumption expenditures as unemployed individuals can expect to substitute wealth with a higher labor income after having found a job. Relaxing the anxiety of having to rely on a low unemployment benefit thus allows a more "optimistic spending" and has the potential to reduce consumption growth for unemployed individuals.

The last terms in (7) and (9) (tautologically) represent the discrete jumps in the level of consumption whenever the employment status actually changes. We will understand more about these jumps after the phase-diagram analysis below.

3.2 Factor rewards and aggregate employment

Workers find markets ("islands") with an infinite supply of jobs with arrival rates μ. Once a market is found, there is perfect competition and factor rewards are given by marginal productivities as in Lucas and Prescott (1974). Firms rent capital on a spot market and choose an amount such that marginal productivity equals the rental rate $w^K(t)$. At the aggregate level, this fixes capital returns r and the gross wage $w(t)/(1-\tau(t))$ at

$$w^K(t) = \partial Y(K(t), L(t))/\partial K(t), \quad w(t)/(1-\tau(t)) = \partial Y(K(t), L(t))/\partial L(t). \tag{11}$$

In this view, there are no vacancies and no cost for vacancies.

An alternative would consist in assigning also vacancies randomly to markets and let wages equal the value of alternative income if the number of workers exceeds the number of vacancies on an island (Shimer, 2007). As this paper builds on the view that workers have a higher labour income than the unemployed, one could extend the Shimer model by allowing for e.g. efficiency wage considerations that set w above alternative income b. This would provide a framework with (almost) perfect competition and vacancies.

Given the job separation and matching setup and assuming a workforce of exogenous size N, (expected) employment at $\tau > t$ is given by

$$L(\tau) = \frac{\mu}{\mu+s}N + \left(L(t) - \frac{\mu}{\mu+s}N\right)e^{-(\mu+s)(\tau-t)} \tag{12}$$

where $L(t)$ is employment today in t. Note that this equation involves two steps: First, we compute expected employment in τ, which gives the right-hand side of this equation. Second, we assume that with a large number of workers, the expected number equals the actual number of workers. It is clear from this equation that employment is basically exogenous, i.e. only a function of time τ. In an aggregate steady state, employment is constant and given by

$$L = \frac{\mu}{\mu+s}N. \tag{13}$$

3.3 Equilibrium

It is useful to distinguish between macro- and micro-variables. There is a deterministic macro level where capital $K(t)$ and aggregate consumption $C(t)$ approach some steady state (K^*, C^*). This is qualitatively similar to the solution of a Ramsey-Solow model without unbounded technological progress and population growth.

All uncertainty takes place at the micro level. Intuitively, optimal consumption functions are determined as follows: With an initial condition for the labour market status z, transitions between employment and unemployment are described by (1). The evolution of wealth follows the budget constraint (3), given an initial condition a_0. Optimal consumption is described by the Keynes-Ramsey rules (7) and (9), depending on the labour market status. The initial condition for consumption needs to be determined such that an intertemporal budget constraint hold. The jump of consumption after a transition in z needs to satisfy such an intertemporal budget constraint as well.

With this understanding of the aggregate level and the consumption functions, one can easily formulate the following

Definition 1 *A competitive equilibrium is described by two paths for $\tau \geq t$ for the aggregate capital stock and aggregate consumption $\{K(\tau), C(\tau)\}$, by paths for employment $L(\tau)$, factor rewards $w(\tau)$, $w^K(\tau)$ and the tax rate $\xi(\tau)$, two functions $c(a, w)$ and $c(a, b)$ and a wealth density $p(a, \tau)$ such that*

1. $K(\tau)$ and $C(\tau)$ start from some initial K_t and C_t such that they approach the steady state (K^*, C^*) on a continuous trajectory
2. $L(\tau)$ is given by (12)
3. given an exogenous path for $b(\tau)$, the government budget constraint (2) and the first-order condition for labour in (11) jointly fix the tax rate $\xi(\tau)$ and wage rate $w(\tau)$ and $w^K(\tau)$ satisfies the first-order condition for capital in (11)
4. the consumption functions $c(a, z)$ satisfy the Keynes-Ramsey rules (7) and (9) plus two boundary conditions to be derived below
5. individual wealth adds up to aggregate wealth as in the first equation of (6) and the density of wealth $p(a, \tau)$ has a mean such that the second equality in (6) is satisfied.

4 Consumption and wealth dynamics

This section characterizes properties of optimal behaviour, taking factor rewards as given. We will return to endogenous factor rewards and thereby to general equilibrium in the subsequent section. We will also assume as of here that all aggregate variables are constant. The appendix discusses how our analysis and proofs can be extended for non-constant aggregate variables.

In a first step, we establish the central link between the interest rate and consumption and savings dynamics. In the second step, we use these findings and combine them with zero-motion lines for wealth to obtain a phase-diagram for consumption and wealth for the employed and unemployed workers. We then prove a theorem on the maximum debt level and thereby complete the characterization of optimal behaviour.

4.1 Consumption growth and the interest rate

- Preliminaries

We first focus on individuals in periods between jumps. The evolution of consumption is then given by the deterministic part, i.e. the dt part, in (7) and (9). We then easily understand

Proposition 1 *Individual consumption rises if and only if current consumption relative to consumption in the other state is sufficiently high.*

For the employed worker, consumption rises if and only if $c(a_w, w)$ relative to $c(a_w, b)$ is sufficiently high,

$$\frac{dc(a_w, w)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_w, b))}{u'(c(a_w, w))} \geq 1 - \frac{r - \rho}{s}. \tag{14}$$

For the unemployed worker, consumption rises if and only if $c(a_b, b)$ relative to $c(a_b, w)$ is sufficiently high,

$$\frac{dc(a_b, b)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_b, w))}{u'(c(a_b, b))} \geq 1 - \frac{r - \rho}{\mu}. \tag{15}$$

Proof. Look at (7) and (9) for $dq_s = dq_\mu = 0$ and note the negative second derivative for instantaneous utility. For details, see app. A.2. ■

- Results

As the conditions in prop. 1 show, saving and wealth dynamics crucially depend on how high the interest rate is. We therefore subdivide our discussion into three parts with r lying in the three ranges given by $(0, \rho]$, $(\rho, \rho + \mu)$, $[\rho + \mu, \infty)$. Before we describe results, let us be clear about one assumption we make for our theoretical analysis.

Assumption 1 *Relative consumption falls in wealth, $\frac{d}{da}\frac{c(a,w)}{c(a,b)} < 0$.*

Starting with the third range $[\rho + \mu, \infty)$, we obtain

Proposition 2 *For a high interest rate, i.e. if $r \geq \rho + \mu$, consumption of employed and unemployed workers always increases.*

Proof. Part (i) can directly be seen from the first expression in (14). As long as $r > \rho$ and $c(a, w) > c(a, b)$, the condition is fulfilled. Part (ii) can also most easily be seen from the first expression in (15). As the smallest value $\frac{c(a,b)}{c(a,w)}$ can take is zero, the term in brackets is always smaller than one. Hence, $r \geq \rho + \mu$ is enough to guarantee that this condition holds. ■

As in other setups with growing consumption, we need to make sure that consumption does not grow too fast. If it does, utility grows too fast and the expected value of the integral in the objective function (4) is not finite. Optimization would then be more involved, which we want to avoid. We therefore have to impose a boundedness condition which implies an upper limit on the interest rate. This condition is derived later in (20) and reads $(1-\sigma)\,r < \rho$. Can such a boundedness condition hold in this high interest rate case where $r \geq \rho + \mu$? It holds if $\mu < \frac{\sigma}{1-\sigma}\rho$. This condition on μ needs to be taken into account in any quantitative analysis.

The high interest rate case reminds of the standard optimal saving result in deterministic setups. If the interest rate is only high enough, consumption and wealth increase over time. This is true here as well. The only difference consists in the fact that the interest rate must be higher than the time preference rate *plus* the job arrival rate.

It is interesting already at this stage to note that the difference for the interest rate as compared to deterministic models is quite substantial. In deterministic models, the interest rate must be larger than the time preference rate. As the job arrival rate is around four times higher than the time preference rate, the interest rate must be much higher here to guarantee wealth growth in all employment states.

The second results is summarized in

Proposition 3 *If the interest rate is at an intermediate level, i.e.* $\rho < r < \rho + \mu$,

(i) consumption of employed workers always increases.

(ii) consumption of an unemployed worker increases only if she is sufficiently wealthy, i.e. if her wealth a_b *exceeds the threshold level* a_b^*, *where the threshold level is implicitly given by*

$$\frac{u'\left(c\left(a_b^*, w\right)\right)}{u'\left(c\left(a_b^*, b\right)\right)} \equiv 1 - \frac{r-\rho}{\mu}. \tag{16}$$

Consumption decreases for $a < a_b^*$.

(iii) At the threshold level a_b^*, *consumption of employed workers exceeds consumption of unemployed workers.*

Proof. With the same line of reasoning as above, condition (14) for the employed worker still holds for $r > \rho$ and therefore consumption of the employed worker always rises. For the unemployed worker, consumption can rise or fall. Imaging $r = \rho + \mu/2$. Then, according to (15), consumption rises only if $\frac{c(a,w)}{c(a,b)} \leq 2^{1/\sigma}$, i.e. if relative consumption is sufficiently small. Given assumption 1, this is the case if a is sufficiently large. Part (iii) simply follows from inserting the condition $\rho < r < \rho + \mu$ into (16). ■

This proposition points to the central new insight for Keynes-Ramsey rules. For the employed worker, the result from deterministic worlds survives: If the interest rate is higher than the time preference rate, consumption and wealth rise. For the unemployed worker, however, this is not true. Consumption and wealth rise only if the unemployed worker is sufficiently rich.

Finally, we have

Proposition 4 *If the interest rate is low, i.e.* $0 < r \leq \rho$, *(i) consumption of employed workers increases if the worker owns a sufficiently low wealth level, $a < a_w^*$. The threshold level a_w^* is given by*

$$\frac{u'(c(a_w^*, b))}{u'(c(a_w^*, w))} \equiv 1 - \frac{r-\rho}{s}. \tag{17}$$

Wealthy workers with $a > a_w^*$ choose falling consumption paths.

(ii) Consumption of unemployed workers always decreases.

(iii) Consumption of employed workers exceeds consumption of unemployed workers at the threshold a_w^*,

$$c(a_w^*, b) = \psi c(a_w^*, w) \tag{18}$$

where $\psi \equiv (u')^{-1}\left(1 - \frac{r-\rho}{s}\right) < 1$.

Proof. In analogy to the previous proof. ■

In deterministic settings, an interest rate of $r \leq \rho$ typically leads to decreases in the level of consumption as the individual does not receive a sufficiently high compensation for her impatience. This classic result survives here for the unemployed worker. The situation looks much better (in a sense) for the employed worker. His wealth level is not run down but increases up to a threshold a_w^*. It decreases only for rich employed workers.

4.2 The reduced form

Before we can derive further properties of optimal behaviour, we need a "reduced form" for optimal behaviour of individuals. We need the smallest number of equations which, once solved, determine an identical number of endogenous variables and which allow us to derive all other endogenous variables subsequently. When looking for such a reduced form, we can always exploit the convenient fact that Poisson uncertainty allows to divide the analysis of a system into what happens between jumps and what happens when a jump takes place. Between jumps, the system evolves in a deterministic way - but does of course take the possibility of a jump into account (as is clearly visible in the precautionary savings terms in the Keynes-Ramsey rules (7) and (9)).[3]

The most convenient way to obtain such a reduced form is to first focus on the evolution between jumps and to eliminate time as exogenous variable. Computing the derivatives of consumption with respect to wealth in both states and considering wealth as exogenous, we obtain a two-dimensional system of non-autonomous ordinary differential equations (ODE). As wealth is now the argument for these two differential

[3] One could be tempted to think of the deterministic parts of the two Keynes-Ramsey rules (7) and (9), jointly with two budget contraints from (3), one for $z = w$ and one for $z = b$, to provide such a reduced form. With an initial condition for wealth and the consumption levels in the different states, one could think of the evolution between jumps as being described by four ordinary differential equations. When solving these equations (conceptionally or numerically), the solution in t for consumption of, say, the unemployed, $c\left(b, a^b(t)\right)$ from (9) would not correspond to consumption $c(b, a^w(t))$ as required in the precautionary savings part in KRR (7) for the employed as wealth levels are accumulated at different speed, i.e. $a^b(t)$ generally differs from $a^w(t)$.

equations, there is no longer a need to distinguish between wealth of employed and unemployed workers. We simply ask how wealth changes in one or the other state given a certain wealth level a. Between jumps, the reduced form therefore reads

$$-\frac{u''(c(a,w))}{u'(c(a,w))}\frac{dc(a,w)}{da}=\frac{r-\rho+s\left[\frac{u'(c(a,b))}{u'(c(a,w))}-1\right]}{ra+w-c(a,w)}, \tag{19a}$$

$$-\frac{u''(c(a,b))}{u'(c(a,b))}\frac{dc(a,b)}{da}=\frac{r-\rho-\mu\left[1-\frac{u'(c(a,w))}{u'(c(a,b))}\right]}{ra+b-c(a,b)}. \tag{19b}$$

With two boundary conditions, this system provides a unique solution for $c(a,w)$ and $c(a,b)$. Once solved, the effect of a jump is then simply the effect of a jump of consumption from, say, $c(a,w)$ to $c(a,b)$.

4.3 A solution for large wealth levels

Let us now consider the CRRA utility function from (5). One property of this reduced form is summarized in the following

Proposition 5 *For any arbitrary fixed parameters ζ_w and ζ_b and for $g\equiv(r-\rho)/\sigma$, system (19) with CRRA preferences (5) is solved by $c(a,z)=(r-g)(a+\zeta_z)$ for large a.*

Proof. Insert this linear solution in (19) using (5). For details, see app. A.3. ■

This proposition says that for sufficiently high wealth levels, labour income is negligibly small. We will exploit this fact and consider very wealthy workers to live in a deterministic world. In fact, we treat them as if they had only capital income and receive unemployment benefits b. We can then easily obtain a boundedness condition (see app. A.4)

$$\rho>(1-\sigma)r, \tag{20}$$

and specify the consumption level as

$$c=(r-g)(a+b/r). \tag{21}$$

Note that any other choice for $\zeta_z\neq 0$ would work as well (and would also satisfy an intertemporal budget constraint including a no-Ponzi game condition). We could use the present value of expected income, $(sb+\mu w)/(r[s+\mu])$, the present value of permanently having a job, w/r, or give high wealth individuals no labour income at all. The behaviour in the limit $a\to\infty$ will be important for the phase diagram analysis. As it will become clear below, the choice of ζ_z is of no importance.

A previous version of this paper allowed individuals to make a labour leisure choice with fixed utility costs ϕ of working in the tradition of Rogerson (1988) and Nosal and Rupert (2007). This would lead to a threshold wealth level a_ϕ above which individuals would prefer to receive unemployment benefits b rather than work and suffer disutility ϕ. This level a_ϕ would be an decreasing function of ϕ. Above this level, workers would also live in a deterministic world and consume $c=(r-g)(a+b/r)$. Rather than extending the model for this labour leisure choice and provide a microfoundation for (21), we leave the maximization problem in its pure form and defend (21) as one example for ζ_z where others would work as well.[4]

4.4 Phase diagram and policy functions

Given the findings on consumption in the above propositions and our reduced form in (19) with its linear solution for large wealth levels in prop. 5, we can now describe the link between optimal consumption and wealth of unemployed and employed workers. We will focus on the intermediate interest rate case first and then present briefly results for the case of low interest rates. The case of high interest rates is the most trivial one with consumption and wealth rising in both employment states. It is therefore not analyzed in detail.

The objective of this section consists in finding consumption paths $c(a, w)$ and $c(a, b)$ which are consistent with optimality conditions obtained so far. The analysis here is more heuristic, a formal existence proof is provided in Bayer and Wälde (2009).

4.4.1 Intermediate interest rate case

- Laws of motion

Let us first represent laws of motion by the usual arrows in fig. 1. The horizontal axis shows wealth a, the vertical axis plots consumption. The zero-motion line for wealth a_b of unemployed workers directly follows from her budget constraint (3). The zero-motion line for consumption $c(a, b)$ is given by (16) in prop. 3. Given policy functions $c(a, w)$ and $c(a, b)$, the condition (16) defines a wealth level a_b^* and thereby the vertical zero-motion line for $c(a, b)$. The solid arrows then show the directions trajectories will take in the four areas delineated by the zero-motion lines for the unemployed.

For the employed workers, there are only two areas. The zero-motion line for wealth, also following from (3) with $z = w$ and drawn as the dashed line, divides the wealth-consumption space into a higher region where wealth a_w falls (as consumption is too high) and a lower region where wealth rises. In both regions, consumption $c(a, w)$ rises as prop. 3 has shown. The dashed arrows show this accordingly.

[4] See app. A.5 for the maximisation problem with a labour supply choice and a discussion of the equivalence of this setup with the one used here.

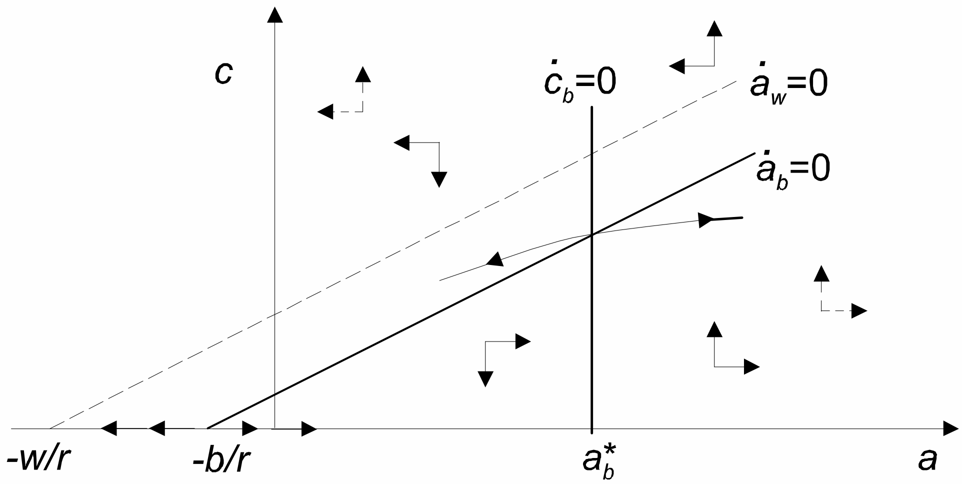

Figure 1 *Components of a phase diagram for wealth and consumption of the unemployed*

This figure shows us that the threshold level a_b^* also provides us with something similar to a steady state. At the intersection point $(a_b^*, ra_b^* + b)$ of the two zero-motion lines, both consumption $c(a, b)$ and a_b do not change. We call this point *temporary* steady state (TSS) for two reasons. On the one hand, unemployed workers experience no change in wealth, consumption or any other variable when at this point (as in a standard steady state of a deterministic system). On the other hand, the expected spell in unemployment is finite and a random transition into employment will eventually shift them out. Hence, the current state is steady only temporarily.

Given the arrow-pairs, we can draw an increasing trajectory $c(a, b)$ through the TSS point with falling consumption and wealth to the left and rising consumption and wealth to the right.

- The borrowing limit

We can complete the phase diagram if we take into account that there is a natural upper limit for debt of an unemployed worker as shown by the following

Proposition 6 *Any individual with initial wealth $a \geq -b/r$ will never be able to or willing to borrow more than $-b/r$. Consumption of an unemployed worker at $a = -b/r$ is zero,*

$$c(-b/r, b) = 0.$$

Proof. "willing to": An employed individual with $a \geq -b/r$ will increase wealth for any wealth levels below a_w^* from (17). If a_w^* is larger than $-b/r$ - which we can safely assume - employed workers with wealth below a_w^* increase wealth and are not willing to borrow more than $-b/r$.

"able to": Imagine an unemployed worker had wealth lower than $-b/r$. Even if consumption is equal to zero, wealth would further fall, given that $\dot{a} = ra + b < 0 \Leftrightarrow a < -b/r$. If an individual could commit to zero consumption when employed and if the separation rate was zero, the maximum debt an individual could pay back is $-w/r$. Imagine an unemployed worker succeeded in convincing someone to lend her "money" even though current wealth is below $-b/r$. Then, with a strictly positive probability, wealth will fall below $-w/r$ within a finite period of time. Hence, anyone lending to an unemployed worker with wealth below $-b/r$ knows that not all of this loan will be paid back. This can not be the case in our setup with one riskless interest rate. Hence, the maximum debt level is b/r and consumption is zero at $a = -b/r$ for an unemployed worker. ■

- The policy functions

We can now bring these properties together and obtain the optimal consumption trajectories $c(a, b)$ and $c(a, w)$. Consider first the situation for wealth below the threshold level a_b^*. Given the arrow-pairs in fig. 1 and the borrowing limit $-b/r$, one can imagine a trajectory $c(a, b)$ leading from the TSS to the borrowing limit with zero consumption. From prop. 4 (iii) and 1, there is a higher trajectory $c(a, w)$ for employed workers. Trajectories (and zero-motion lines) for employed workers are always represented by dashed curves, as drawn. Relative consumption in the TSS is given by (16). Arrow pairs for employed workers are satisfied if this trajectory lies below the zero-motion line for a_w.

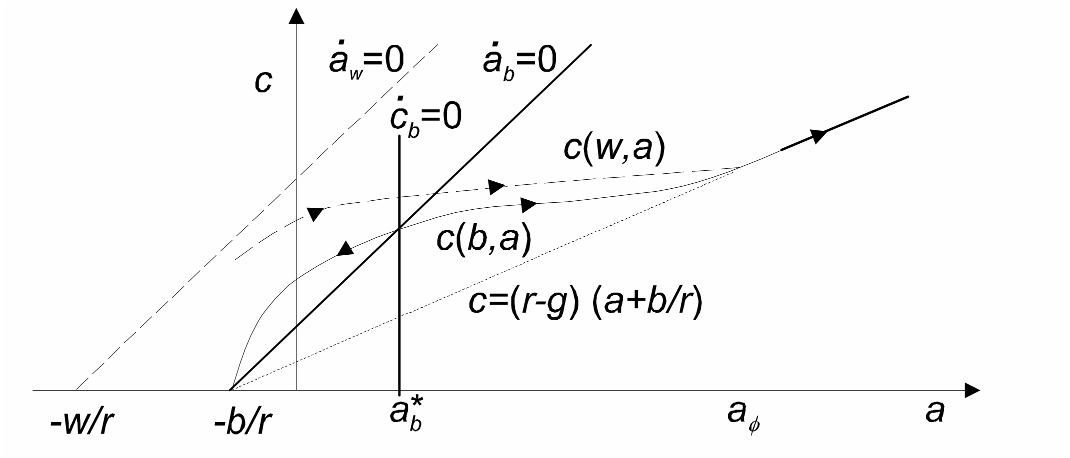

Figure 2 *Policy functions for employed and unemployed workers (intermediate interest rate)*

Let us now proceed with higher wealth levels. We start with wealth levels which are "sufficiently large" in the sense of prop. 5. Let us denote this large level for illustration purposes by a_ϕ. For $a \geq a_\phi$, household consumption increases linearly in wealth. In the above figure, this is the line to the right of a_ϕ and going north-east.

This line is consistent with arrow pairs as long as it is below both zero-motion lines for wealth. We assume for this figure that the arbitrary parameter ζ_i in prop. 5 is given by b/r as in (21). This would be the consumption level of an unemployed worker who knows that she will remain unemployed forever.

Linking the trajectories $c(a, w)$ and $c(a, b)$ for $a < a_b^*$ with the linear consumption line for $a \geq a_\phi$ then provides a complete optimal consumption rule in both labour market states. We have thereby found consumption paths which are consistent with optimal behaviour.

- Properties of optimal behaviour

When the wealth level is lower than a_ϕ but higher than a_b^*, workers accumulate wealth in both employment states. The level of consumption of the unemployed worker is always higher than in a situation where labour income is permanently at b. The worker anticipates occasional jumps into employment and smooths consumption by increasing the consumption level already while unemployed. This follows formally from the fact that the TSS $(a_b^*, c(a_b^*, b))$ for the unemployed worker lies above the consumption line where labour income is permanently given by b and that the policy function $c(a, b)$ must go through this point. The latter must hold as otherwise the statement of proposition 3 on negative savings below a_b^* for unemployed worker would be contradicted.

We can now analyse the effects of job losses or job acceptance. Such an analysis is straightforward as the effects of a jump is simply given by a jump from one consumption line to the other. Assume a worker is employed and starts with a wealth level of zero. Then savings are positive and wealth rises. When he loses the job, consumption drops from $c(a, w)$ to $c(a, b)$. If the wealth level at this moment is smaller than a_b^*, savings become negative and consumption falls. If wealth is higher than a_b^*, savings remain positive, despite the loss in labour income, and consumption continues to rise.

The new insight of this analysis is clearly the existence of threshold levels a_w^* and a_b^*. Fig. 2 allows to compare the saving behaviour of an unemployed worker with income b who knows that he will never find a job and an unemployed worker who has the hope to find one (i.e. there is a positive arrival rate). The unemployed worker who knows that he will eventually find a job anticipates this effect and increases the consumption level. This higher consumption level implies, however, that there needs to be sufficiently high capital income to make sure that savings are positive. If wealth is too low, anticipating higher future labour income implies a reduction in wealth. "Postcautionary dissaving" takes place.

4.4.2 Low interest rate

- Laws of motion and policy functions

Let us now analyse the case of a low interest rate. Proposition 4 shows that consumption of unemployed workers falls for all wealth levels and that consumption of employed workers increases only for wealth levels below a threshold level a_w^*. A second obvious difference to the case before is the slope of the consumption line for

$a \geq a_\phi$. It is now higher than the slope of the zero-motion lines as $g = (r - \rho)/\sigma$ is negative.

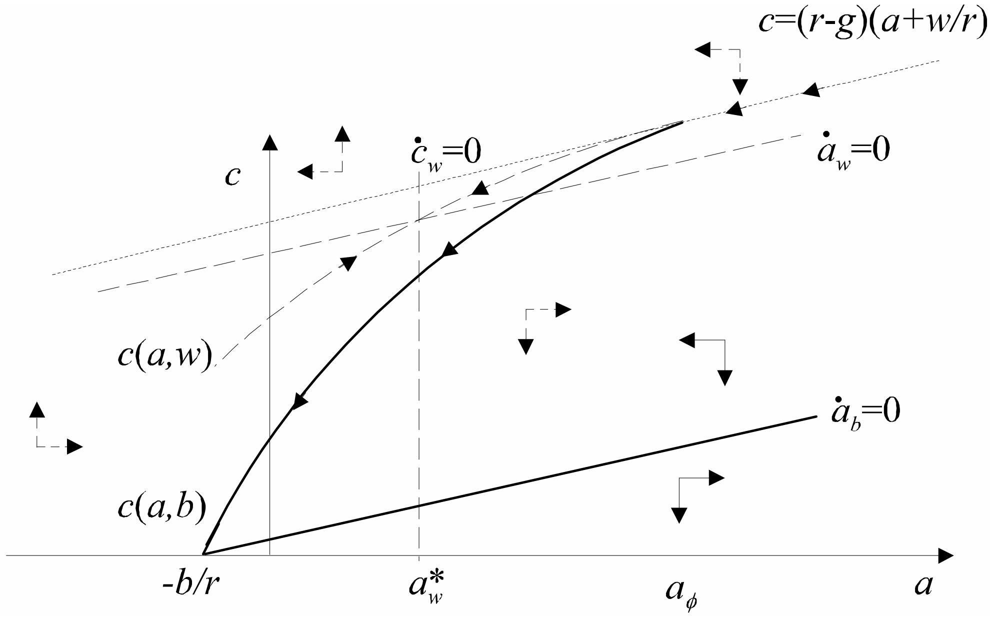

Figure 3 *Policy functions for employed and unemployed workers (low interest rate)*

Fig. 3 shows dashed zero-motion lines for a_w and $c(a, w)$ and the TSS

$$\Theta \equiv (a_w^*, c(a_w^*, w)) \tag{22}$$

for the employed workers (in contrast to the TSS for unemployed workers in fig. 2) at the intersection point. The zero-motion line for a_b is also plotted. As we know from prop. 4 that consumption for the unemployed falls, we know that above the zero-motion line for a_b, consumption and wealth fall for the unemployed. The arrow-pairs for the employed workers are also added. They show that one can draw a saddle-path through the TSS. To the left of the TSS, wealth and consumption of employed workers rise, to the right, they fall.

Relative consumption when the employed worker is in the TSS is now given by (17). A trajectory going through $(a_w^*, c(a_w^*, b))$ and hitting the zero-motion line of a_b at $-b/r$ is in accordance with laws of motions for the unemployed worker.

By ass. 1 relative consumption falls when wealth is higher. At a_ϕ, which should be thought again of being very high or infinity, relative consumption is identical and consumption increases linearly in wealth.

- Properties of optimal behaviour

The case of a low interest rate is particularly useful as the range of wealth a worker can hold is bounded. Whatever the initial wealth level, the wealth level will be in the range $[-b/r, a_w^*]$ after some finite length of time. This follows directly from the policy functions in fig. 3. Once within the range $[-b/r, a_w^*]$, wealth will increase while employed and decreased while unemployed. A worker will be in a permanent consumption and wealth cycle due to precautionary savings.

It is clear that this low-interest rate case is the primary candidate for a stationary state in an aggregate equilibrium. One can easily imagine a distribution of wealth over the range $[-b/r, a_w^*]$ whose mean times the number of workers would then give the aggregate capital stock.

4.5 Existence of equilibrium

All steps undertaken so far were explorative in the sense that no proof for the existence of a path $c(a, z)$ has been given. Existence for the low interest rate case depicted in fig. 3 can be proven, however, at least under a mild technical condition.

In fig. 3, we implicitly considered solutions of our system in the (open) set $Q = \{-b/r \leq a\} \cap \{c(a, w) \leq ra + w\} \cap \{ra + b \leq c(a, b)\} \cap \{0 \leq c(a, b)\} \cap \{c(a, b) \leq c(a, w)\}$. In words, consumption of the employed worker is below the zero-motion line for her wealth, consumption of the unemployed worker is above her zero-motion line for wealth, consumption of the unemployed worker is positive and wealth is higher than the maximal debt level b/r. Given some lemmas in Bayer and Wälde (2009, sect. 3), we also know that consumption of employed workers always exceeds consumption of unemployed workers. For the existence proof we restrict this set to

$$\begin{aligned} R_{v,\Psi} = \big\{ & (a, c(a, w), c(a, b)) \in \mathbb{R}^3 | \, (a, c(a, w), c(a, b)) \in Q, \\ & c(a, w) \leq \Psi < \infty, \, a \leq (\Psi - w + v)/r \big\}, \end{aligned} \tag{23}$$

where Ψ is a finite large constant. There are two differences to Q : First, the set $R_{v,\Psi}$ is bounded. This is a purely technical necessity. Second, the set $R_{v,\Psi}$ excludes the zero-motion line for wealth a_w by subtracting a small positive number v. We need to do this as the fraction on the right-hand side of our differential equation (19a) is not defined for the TSS.[5]

We now introduce an auxiliary TSS (aTSS) in order to capture v. In analogy to the TSS Θ from (22), this point is defined by

$$\Theta_v \equiv (a_w^*, c_v(a_w^*, w)),$$

i.e. the wealth level a_w^* is unchanged but the consumption level is "a bit lower" than in the TSS. In the TSS, the consumption level is on the zero-motion line, i.e. $c(a_w^*, w) = ra_w^* + w$. In the auxiliary TSS, the consumption level is on the line $ra + w - v$ and therefore given by $c_v(a_w^*, w) = ra_w^* + w - v$. Let us now consider the following

Definition 2 *(Optimal consumption path) An optimal consumption path is a solution $(a, c(a, w), c(a, b))$ of the ODE-system (19) for the range $-b/r \leq a \leq a_w^*$ in $R_{v,\Psi}$ with terminal condition $(a_w^*, c_v(a_w^*, w), c_v(a_w^*, b))$ which satisfies $c(-b/r, b) = 0$. In analogy*

[5] While this is a standard property of many steady states, the standard solutions (e.g. linearization around the steady state) do not work in our case. This is in part due to the fact that the original stochastic differential equation system (see 7 as an example) is a delay differential equation system. See the conclusion of Bayer and Wälde (2010) for an outlook.

to the aTSS and to (18), the terminal condition satisfies $c_v(a_w^*, w) = ra_w^* + w - v$ *and* $c_v(a_w^*, b) = \psi c_v(a_w^*, w)$.

Bayer and Wälde (2009, sect. 3) then prove

Theorem 1 *There is an optimal consumption path.*

This establishes that we can continue in our analysis by taking the existence of a path $c(a, z)$ as given. Intuitively speaking, i.e. looking at v as very small constants close to zero, we know that there are paths $c(a, w)$ and $c(a, b)$ as drawn in fig. 3.

5 The distribution of labour income and wealth

5.1 Labour market probabilities

Consider first the distribution of the labour market state. Given that the transition rates between w and b are constant, the conditional probabilities of being in state $z(\tau)$ follow e.g. from solving Kolmogorov's backward equations as presented in Ross (1993, ch. 6). As an example, the probability of being employed in $\tau \geq t$ conditional on being in state $z \in \{w, b\}$ in t are

$$P(z(\tau) = w | z(t) = w) \equiv p_{ww}(\tau) = \frac{\mu}{\mu + s} + \frac{s}{\mu + s} e^{-(\mu+s)(\tau - t)}, \tag{24}$$

$$P(z(\tau) = w | z(t) = b) \equiv p_{bw}(\tau) = \frac{\mu}{\mu + s} - \frac{\mu}{\mu + s} e^{-(\mu+s)(\tau - t)}. \tag{25}$$

The complementary probabilities are $p_{wb}(\tau) = 1 - p_{ww}(\tau)$ and $p_{bb}(\tau) = 1 - p_{bw}(\tau)$. The unconditional probability of being in state z in τ is given by

$$p_z(\tau) = p_w(t) p_{wz}(\tau) + (1 - p_w(t)) p_{bz}(\tau) \tag{26}$$

where $p_w(t)$ is the probability of $z(t) = w$, i.e. describes the initial distribution of $z(t)$.

5.2 Fokker-Planck equations for wealth

- Some background

Now consider one individual with a level of wealth of $a(t)$ and an employment status $z(t)$. This individual faces an uncertain future labour income stream $z(\tau)$. What is the joint distribution of $a(\tau)$ and $z(\tau)$ for $\tau > t$? Using methods from stochastic, we can compute the Fokker-Planck equations. They describe the evolution of the (joint) density of $(a(\tau), z(\tau))$, i.e. of the labour market status and wealth for $\tau \geq t$. This density is denoted by $p(a, z, \tau)$ and obviously driven by a discrete and a continuous random variable. We can therefore split it into two "subdensities"

$p(a,w,\tau)$ and $p(a,b,\tau)$ which can be understood as the product of a conditional probability times the probability of being in a certain employment state,

$$p(a,z,\tau) \equiv p(a,\tau|z)\, p_z(\tau) \tag{27}$$

The probability $p_z(\tau)$ of an individual to be in a state z in τ is given by (26). The conditional density of $a(\tau)$ given $z(\tau)$ is denoted by $p(a,\tau|z)$.

Note that the distribution of $(a(\tau), z(\tau))$ certainly depends on the initial condition $(a(t), z(t))$, which needs to be specified in order to calculate $p(a,z,\tau)$. In the notation we do not distinguish between the following two possibilities. Firstly, $(a(t), z(t))$ can be deterministic numbers, in which case $p(a,z,t)$ is a Dirac-distribution centered in $(a(t), z(t))$ (more precisely, $a \to p(z(t), a, t)$ is a Dirac-distribution). Secondly, $(a(t), z(t))$ can itself be random, either because we regard them as outcomes of the employment-wealth-process started at an even earlier time, or because we might consider a population with certain employment probabilities and wealth distributions (see below in sect. 6).

As is clear from (27), $p(a,z,\tau)$ are not conditional densities - they rather integrate to the probability of $z(\tau) = z$. Looking at an individual who is in state z in τ, we get

$$\int p(a,z,\tau)\, da = \int p(a,\tau|z)\, p_z(\tau)\, da = p_z(\tau) \int p(a,\tau|z)\, da = p_z(\tau). \tag{28}$$

The density of a at some point in time τ for of an individual with initial condition $(a(t), z(t))$ is then simply

$$p(a,\tau) = p(a,w,\tau) + p(a,b,\tau). \tag{29}$$

See fig. 4 for an illustration of these (sub-) densities and how they qualitatively look like at an arbitrary point in time $\tau \geq t$.

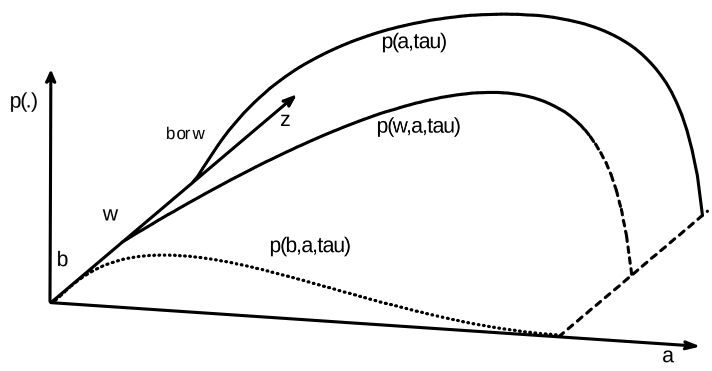

Figure 4 *The subdensities $p(a,b,\tau)$ and $p(a,w,\tau)$ and the density $p(a,\tau)$*

- The equations

The derivation of the Fokker-Planck equations is in Bayer and Wälde (2009, sect. 4). The result is a system of two one-dimensional linear partial differential equations in $p(a, w, \tau)$ and $p(a, b, \tau)$,

$$\begin{aligned} &\frac{\partial}{\partial \tau} p(a, w, \tau) + \{ra + w - c(a, w)\} \frac{\partial}{\partial a} p(a, w, \tau) \\ &+ \left\{r - \frac{\partial}{\partial a} c(a, w) + s\right\} p(a, w, \tau) - \mu p(a, b, \tau) = 0, \end{aligned} \tag{30a}$$

$$\begin{aligned} &\frac{\partial}{\partial \tau} p(a, b, \tau) + \{ra + b - c(a, b)\} \frac{\partial}{\partial a} p(a, b, \tau) \\ &+ \left\{r - \frac{\partial}{\partial a} c(a, b) + \mu\right\} p(a, b, \tau) - sp(a, w, \tau) = 0. \end{aligned} \tag{30b}$$

The differential equations are linear with derivatives in τ and a. They are non-autonomous as coefficients of the densities and their derivatives are functions of a. As we can see, the density is linked to optimal behaviour through the consumption levels $c(a, w)$ and $c(a, b)$ (and their partial derivatives, i.e. the marginal propensities to consume out of wealth) obtained from the solution of the individual optimization problem.

Compared to closed-form solutions for transition densities which are used in finance (see e.g. Aït-Sahalia, 2004), our differential equations are of course less informative. The closed form solutions build on linear stochastic differential equations, however. The absence of closed-form solution here is therefore simply the result of the non-linearity of our optimal consumption functions $c(a, z)$.

5.3 Existence and uniqueness of and convergence to a limiting distribution

Let us first note that the limiting distribution for the employment status $z(\tau)$ can easily be seen from (24) and (25) and their complementary probabilities. For any fixed initial condition $z(t) \in \{w, b\}$, the limiting distribution is given by $p(w) = \mu/(\mu + s)$ and $p(b) = s/(\mu + s)$. The same limiting distribution results if the initial condition is given by a distribution itself. Convergence to this limiting distribution can be seen from letting τ in (24) and (25) increase and approach infinity.

The question of existence of and convergence to a limiting distribution for the joint density $p(a, z, \tau)$ is far more involved. Let us introduce the notion of an *invariant distribution*. A distribution for (a, z) is called invariant, if $(a(\tau), z(\tau))$ follows the distribution for any time $\tau > t$ provided that $(a(t), z(t))$ does. Obviously, any limiting distribution must be invariant. Building on the general ergodicity-theory for Markov-processes by Down et al. (1995), we obtain

Theorem 2 *Let $r < \rho$ and assume there is a temporary steady state (TSS). Assume further that the initial distribution of wealth is supported in $[-b/r, a_w^*]$. Then,*

(i) the distribution of wealth at any subsequent time $\tau \geq t$ is supported in $[-b/r, a_w^*]$,

(ii) there is a unique invariant wealth distribution, and

(iii) for any such initial distribution, the distribution of wealth converges to the invariant distribution.

Proof. see Bayer and Wälde (2009, sect. 5) ■

Note that this theorem can be proven for the set R_Ψ only, i.e. the parameter v from the set $R_{v,\Psi}$ in (23) used for the proof of theo. 1 must be zero. This proof therefore requires the existence of a TSS and not of an auxiliary TSS. Note also that if $r > \rho$, no invariant probability distribution and, hence, no limiting distribution exists.

5.4 Some background for numerical solutions

- Initial conditions

Obtaining a unique solution for ODEs generally requires certain differentiability conditions and as many initial conditions as differential equations. Conditions for obtaining a unique solution for PDEs are not as straightforward. One can nevertheless understand easily that initial functions are required to obtain a solution: Let us assume for our case two initial functions for a, one for each labour market state $z \in \{w, b\}$. The obvious interpretation for these initial functions are densities, just as illustrated in fig. 4. Initial functions would therefore be given by $p(a, b, t) = p^{ini}(a, b)$ and $p(a, w, t) = p^{ini}(a, w)$. Clearly, they take positive values on the range $[-b/r, a_w^*]$ only and need to jointly integrate to unity. Given these initial functions, one can then compute the partial derivatives with respect to a in (30). This gives an ODE system which allows us to compute the density for the "next" τ. Repeating this gives us the densities for all z, a and τ we are interested in.

An initial function for wealth in each labour market state sounds unusual when thinking of one individual who, say, in t has wealth of $a(t)$ and is currently employed, $z(t) = w$. One can express these two deterministic numbers such that we obtain initial functions, however. First, $p^{ini}(a, b) = 0$: as the probability for an employed individual to be unemployed is zero and the probability of being unemployed is given by $\int_{-b/r}^{a_w^*} p^{ini}(a, b)\, da$ (compare the example in (28)), $p^{ini}(a, b)$ must be zero. Second, there are two possibilities for $p^{ini}(a, w)$. Either one considers $p^{ini}(a, w)$ as a Dirac-distribution, i.e. there is a degenerate density with mass-point at $a = a(t)$. Or, maybe most convenient both for numerical purposes and for intuition, one considers the current wealth level $a(t)$ to be observed with some imprecision. Pricing various types of assets (cars or other durable consumption goods like a house) might not be straightforward and one can easily imagine an initial function which is zero to the left of $a_{\min}$ and to the right of $a_{\max}$ and condenses all probability between these values (which can of course be arbitrarily close to $a(t)$).

- The long-run distribution of individual wealth

When we are interested in the long-run distribution of wealth and income only, the time derivatives of the densities would be zero and the long-run densities would be described by two linear ordinary differential equations,

$$\{ra + w - c(a, w)\} \frac{\partial}{\partial a} p(a, w) + \left\{ r - \frac{\partial}{\partial a} c(a, w) + s \right\} p(a, w) - \mu p(a, b) = 0, \tag{31a}$$

$$\{ra + b - c(a, b)\} \frac{\partial}{\partial a} p(a, b) + \left\{ r - \frac{\partial}{\partial a} c(a, b) + \mu \right\} p(a, b) - sp(a, w) = 0. \tag{31b}$$

The advantage of these two differential equation systems is clear: if numerical procedures can be found to easily solve them, short-run and long-run distributions can be obtained without having to simulate a system. These equations also open up new avenues for structural estimation. Parameters can easily be estimated such that an observed distribution is optimally fitted by these predicted distributions.

- Boundary conditions for the long-run distribution

For the long-run distribution in (31), two boundary conditions are given by

Proposition 7 *Let the bounds of the range of* a *be given by* $-b/r$ *and* a_w^* *as illustrated in fig. 3. Boundary conditions are provided by*

$$p(a_w^*, w) = 0, \quad p(a_w^*, b) = 0. \tag{32}$$

Proof. Fig. 3 shows that a_w^* can be reached only in state w. a_w^* and the corresponding consumption level are a saddle point which is approached asymptotically. In practice, this level is never reached and $p(a_w^*, w) = 0$. The second boundary condition is then an immediate consequence. As the state (a_w^*, b) can occur only through a transition from (a_w^*, w) but the density at (a_w^*, w) is zero, $p(a_w^*, b) = 0$ as well. ■

6 The aggregate distribution of wealth and employment

Using all the results we collected so far on individual behaviour, we are now in an easy position to describe the aggregate distribution of wealth and employment. One statistic one generally would like to understand is the share of the population which has a wealth below a certain level. The population consists of N individuals. Wealth and labour market status of an individual i is described by the density $p_i(a, z, \tau)$ given an initial condition $(a_i(t), z_i(t))$. The density of each single individual is described by the PDEs in (30). The density of individual wealth (without taking the labour market status into account) is $p(a, \tau)$ from (29).

Now define the share of individuals in the entire population with wealth below a certain level a at some point in time $\tau > t$ as $H(a, \tau) \equiv \Sigma_{i=1}^N I(a_i(\tau)) / N$ where $I(a_i(\tau))$ is the indicator function taking a value of 1 if $a_i(\tau) < a$ and 0 otherwise. As the $a_i(\tau)$ are identically and independently distributed[6], the strong law of large numbers holds and we obtain $\lim_{N\rightarrow\infty} H(a,\tau) = \int_{-b/r}^{a} p(x,\tau)\,dx$. In words, the share of individuals in our population with wealth below a is given by the probability that an individual has wealth below a.[7] Computing the derivative of the distribution function gives the density of wealth for the population as a whole, $h(a,\tau) \equiv \frac{d}{da}\int_{-b/r}^{a} p(x,\tau)\,dx = p(a,\tau)$.

When we are interested in wealth distributions for each labour market status individually, we can define $H(a,z,\tau) \equiv \Sigma_{i=1}^{N} I(a_i(\tau), z(\tau))/N$ where the indicator function takes the value of one if $a_i(\tau) < a$ and $z(\tau) = z$. The density is then given by $h(a,z,\tau) = p(a,z,\tau)$.

As has been stressed in the discussion after (27), the initial condition $(a(t), z(t))$ can itself be random. This means that a solution of (30) with an initial distribution for a and z capturing some real world distribution of wealth and employment status provides a prediction how this aggregate distribution evolves over time. We describe our initial conditions by two subdensities, one for employed individuals and one for unemployed individuals, similar to the subdensities in (29),

$$h(a,w,t) = h^{ini}(a,w), \quad h(a,b,t) = h^{ini}(a,b).$$

Empirical information needed to find plausible initial functions (or to estimate them) is the distribution of wealth for employed and unemployed workers. If the share of unemployed workers is x%, the density $h^{ini}(a,w)$ must integrate to x/100, given the property of the subdensity $p(a,w,\tau)$ as shown in (28). If one is primarily interested in understanding the prediction for the aggregate distribution of wealth, any reasonable functions with range $[-b/r, a_w^*]$ and satisfying (28) will do.

7 Conclusion

The objective of this paper was to extend the standard labour market matching model by allowing individuals to save. Due to the continuous time setup chosen here, various results were obtained which increase intuitive understanding of the basic economic mechanisms.

The Keynes-Ramsey rule for this setup reveals that precautionary savings are at work here. Consumption grows faster while employed (as compared to a situation without labour income fluctuations) and grows slower while unemployed.

[6] Independence could be questioned in a setup where the arrival rate of a job depends on the number of vacancies. As long as the number of vacencies is a deterministic function of time (there is no aggregate uncertainty) and the matching rate μ would then also be a deterministic function of time, independence is preserved. In our aggregate stationary state where all arrival rates are constant, the $a_i(\tau)$ are independent for sure.

[7] It is hard to imagine an economy with an infinite number of agents N. The alternative to this discrete law of large numbers is to work with a continuum of agents of mass N. The concept of infinity is then available by construction and laws of large numbers do not encounter the problem of having to imagine what an infinite number of individuals mean. On the downside, one runs into many well-known technical problems and, maybe more importantly, it might be just as difficult to imagine a continuum of individuals as an economy with an infinite number of inhabitants.

Combining the Keynes-Ramsey rules for both labour market states with the corresponding budget constraints has shown that an equilibrium with a stationary long-run wealth distribution can exist only for a situation where the interest rate is below the time preference rate. For this "low interest rate case", the distribution of wealth is endogenously bounded by the minimum wealth of $-b/r$ and by the maximum wealth level a_w^* an employed worker would want to hold. An employed worker chooses consumption such that consumption and wealth rise over time. Both consumption and wealth approach an upper limit (a_w^*, c_w^*) which was called temporary steady state. An unemployed worker is on an optimal consumption path which reaches the maximum debt level b/r in finite time.

The paper then derived the Fokker-Planck equations which describe the evolution of the distribution of wealth and labour market status for each individual worker starting from some initial condition. It has been shown that there exists a unique long-run distribution and that initial distributions converge to this long-run distribution. The analysis of distributions of labour market status and wealth in an economy with many agents has also been undertaken.

8 Appendix

All proofs are in Bayer and Wälde (2009). All appendices numbered A.1, A.2 etc. are available upon request.

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