Describing the Dynamics of Distributions in Search and Matching Models by Fokker-Planck Equations[1]

Christian Bayer[(a)] and Klaus Wälde[(b)]
[(a)] University of Vienna and
[(b)] University of Mainz and CESifo, University of Bristol, Université catholique de Louvain[2]

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The analysis of distributions is central to search and matching models. Wage distributions are a central concern but distributions of productivities, firm types, human capital, entitlement to unemployment benefits and wealth become increasingly important. We present a method - the Fokker-Planck equations - which allows to describe and analyse the dynamics of distributions in a very general way. We illustrate this approach by analysing optimal saving of risk-averse households in a frictional labour market. Our Fokker-Planck equations describe the evolution of the joint distribution of labour market status and wealth. A very intuitive interpretation is provided.

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1 Introduction

Everybody searches. We search for jobs, for bars, for good food, occasionally even for happiness. We sometimes find what we look for but it always takes time. This is

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of course the fundamental insight which the Diamond-Mortensen-Pissarides (DMP) models have incorporated so successfully in labour economics and beyond.

All of the models in this tradition use stochastic processes as building blocks. The defining process is the one that moves workers stochastically between employment and unemployment. Some of the DMP-type models largely abstract from distributional predictions resulting from these processes, some focus explicitly on distributional properties.

A classic example for the first group is the seminal contribution by Pissarides (1985) and the work inspired by it. As the assumption of a very large number of agents allows to employ a law of large numbers, the description of the economy is basically deterministic. Central variables of interest, e.g. the unemployment rate, can be described by an ordinary differential equation. More precisely speaking, almost all matching models focus on the mean of the underlying stochastic process, but abstract from positive variances which would occur with a finite number of agents. While a law of large numbers is easily acceptable for economy-wide (un-) employment rates, this is less obvious for "large" firms. Most real-world firms have so few employees (especially when taking the heterogeneity of employees into account) that a law of large numbers seems like a relatively strong assumption.

The seminal contribution where endogenous distributions are explicitly taken into account is the Burdett and Mortensen (1998) paper. Individuals move up on a wage ladder, occasionally losing their job and taking an endogenous reservation wage into account. This stochastic process implies a distribution of wages (which is then supported by wage-posting of firms). Subsequent work extends this analysis to allow for structural estimation of the effect of frictions on the distribution of wages (Postel-Vinay and Robin, 2002) and for competition between firms for workers (Cahuc et al. 2006). Moscarini and Postel-Vinay (2008, 2010) and Coles and Mortensen (2011) present variants of the Burdett and Mortensen model to analyse transitional dynamics allowing, inter alia, to understand the evolution of the firm-size distribution over the business cycle.[3] An explicit wage distribution under an uncertain match quality as in Jovanovic (1984) allowing for Bayesian learning is presented in a Mortensen-Pissarides (1994) type equilibrium model by Moscarini (2005).[4]

An additional endogenous distribution arises if some state variable is added. The accumulation of human capital as in Burdett et al. (2011) or wealth as in Shimer and Werning (2007, 2008) or Lise (2010) are some examples in this direction. Thinking of entitlement to unemployment insurance payments and how it is accumulated while employed and reduced while unemployed would be another (see Bontemps et al., 1999, for an analysis of exogenous distributions of entitlement).

Most of these examples can be identified to share a common fundamental structure: There is some fundamental stochastic process (workers moving in and out of employment or up on a wage ladder) which implies a distribution of the variable un-

[3] There is another class of models where distributional aspects play a crucial role as e.g. in the endogenous job destruction model of Mortensen and Pissarides (1994). In models of this type, the distribution is explicitly given and not derived from more fundamental processes.

[4] Wage distributions can also be derived in marriage models e.g. for single females (see Jacquemet and Robin, 2010).

der consideration. In the case of additional state variables like wealth, human capital or entitlement to benefit payments, there is an additional process that also describes the evolution of some density over time.[5]

It is the objective of this paper to provide a tool that embeds the analysis of distributions into a standard mathematical tool - the so-called Fokker-Planck equations. These equations describe the distributional properties of stochastic processes in a fairly general but still intuitive way. The advantage of these equations consists in the fact that one is no longer restricted to specific distributions for which closed-form solutions can be found. The entire dynamics of distributions is described and not simply distributions in a "steady-state". They can also be applied to much more general processes than has been done so far in the literature. By their nature, all existing distributions must be special cases of these general equations.[6]

As a generic example to illustrate this method, we build on the world of the Diamond-Mortensen-Pissarides models and allow for wealth accumulation. It is standard practice in this literature to assume strong capital market imperfections implying that households consume their current income. When households are allowed to save, however, they self-insure against labour market shocks by wealth accumulation allowing for consumption smoothing.[7]

There are various reasons why this particular example of wealth accumulation in a frictional market context is of importance. One can expect that bargaining and labour supply choices are affected by personal wealth. Analysing the effects of labour market policies is probably biased if wealth is not taken into account as wealth should also affect search intensity. Normative analyses of optimal unemployment benefit schemes should also take wealth issues into account as social welfare functions or other optimality criteria neglecting wealth tend to be incomplete from a conceptional perspective.

The big advantage of this example for illustrating the usefulness of Fokker-Planck equations, however, consists in the generic nature of the resulting stochastic system. There will be one fundamental equation that describes the ins into and outs out of employment. Then, there will be one "dependent" equation that describes the

[5] This general description should make clear that the method to be presented here can of course also be applied to directed search setups as in Moen (1997), Acemoglu and Shimer (1999) or Shi (2009). Firm size distributions with aggregate shocks in a directed search setup as in Kaas and Kircher (2011) would be another application where methods presented here promise to simplify the description and analysis of the dynamics of distributions. As another example, take total factor productivity to be the fundamental process and capital the corresponding state variable. The only condition for the applicability is that the model is set in continuous time. In this sense, models building on Sannikov (2007) or Sannikov and Skrzypacz (2008) could fruitfully use Fokker-Planck equations as well.

[6] As Fokker-Planck equations describe densities, this method would allow for structural maximum likelihood estimation of models that include additional features to those usually captured in labour models (see e.g. van den Berg, 1990; Postel-Vinay and Robin, 2002; Flinn, 2006; see also Launov and Wälde, 2010).

[7] Our example used for highlighting the usefulness of Fokker-Planck equations is therefore related to the precautionary-savings literature (Huggett, 1993, Aiyagari, 1994 and subsequent work). For savings in a matching framework, see also the work by Lentz and Tranaes (2005), Lentz (2009) and Krusell et al. (2010).

accumulation of wealth. If wealth is replaced by firm-size, human capital, entitlement to benefits or duration in employment or unemployment, exactly the same structure occurs. We will therefore highlight later in the text how other applications can very easily apply this method.

Before we can derive Fokker-Planck equations, we solve the consumption-saving problem of an individual. Optimal behaviour is described by a generalized Keynes-Ramsey rule where the generalization consists in a precautionary savings term. This term lends itself to intuitive economic interpretation. In a second step, we provide a phase-diagram analysis of the optimal behaviour of an individual, i.e. of the evolution of wealth and consumption when labour income jumps between being high and low. In addition to this illustration, we also provide a formal existence proof for optimal consumption-wealth profiles for both labour market states.

The third step then provides the main contribution of this paper. It inquires into the distributional properties of wealth and labour market status. Using the Dynkin formula, we obtain the Fokker-Planck equations for the wealth-employment status system. We obtain a two-dimensional partial differential equation system. It describes the evolution of the density of wealth and employment status over time, given some initial condition. When we are interested in long-run properties only, we can set time derivatives equal to zero in the Fokker-Planck equations and obtain an ordinary two-dimensional non-autonomous differential equation system. Boundary conditions can be motivated from our phase diagram analysis.[8]

This paper is related to various strands of the literature. The analysis of optimal consumption behaviour builds on earlier work of one of the authors (Wälde, 1999, 2005) who analyzes optimal saving under Poisson uncertainty affecting the return to capital but not labour income.[9] We also use the insights of the long literature using setups with continuous time uncertainty. Starting with Merton (1969), it includes, inter alia, the work of Turnovsky (see e.g. Turnovsky, 2000), Bentolila and Bertola (1990), Bertola et al. (2005) and Shimer and Werning, (2007, 2008).

The principles behind and the derivation of the Fokker-Planck equation (FPE) for Brownian motion are treated e.g. in Friedman (1975, ch. 6.5) or Øksendal (1998, ch. 8.1). For our case of a stochastic differential equation driven by a Markov chain, we use the infinitesimal generator as presented e.g. in Protter (1995, ex. V.7). From general mathematical theory, we know that the density satisfies the corresponding FPE $\frac{\partial}{\partial t}p(t,x) = \mathcal{A}^* p(t,x)$, where p denotes the density of the process with state variable x at time t and $\mathcal{A}^*$ is the adjoint operator of the infinitesimal generator $\mathcal{A}$ of this process. We follow this approach in our framework and obtain the FPE for the law of the employment-wealth process.

In economics, versions of Fokker-Planck equations (also called Kolmogorov forward equations) are rarely used or referred to so far. Papers we are aware of are Lo (1988),

[8]Existence and uniqueness of a stationary distribution of wealth and labour market status and convergence to this distribution is proven in a companion paper (Bayer and Wälde, 2011).

[9]Work completed before the present paper includes an unpublished PhD dissertation by Sennewald (2006) supervised by one of the authors which contains the Keynes-Ramsey rules. Toche (2005) considers the saving problem of an individual where job-loss is permanent and unemployment benefits are zero. Lise (2006) developed a Keynes-Ramsey rule for times between jumps as well.

Merton (1975), Klette and Kortum (2004), Moscarini (2005), Koeniger and Prat (2007) and Prat (2007). Lo derives a FPE for a one-dimensional process. Merton applies the method to analyse distributional properties of a stochastic Solow growth model. Klette and Kortum employ a method related to FPEs to derive firm-size distributions. Moscarini uses them to derive the distribution of the belief about the quality of a match. Koeniger and Prat obtain an employment distribution and Prat describes the distribution of detrended productivity.

The main difference in our application consists in its considerable generalization, in the detailed derivation and in the explanations linking the derivation to standard methods taught in advanced graduate courses. The only new tool we require and which we introduce intuitively is the Dynkin formula. This approach focusing on the principles of FPEs in a tractable and accessible way should allow and encourage a much wider use of this tool for other applications. We would like to move Fokker-Planck equations much more into the mainstream. In fact, one could argue that Fokker-Planck equations should become a tool as common as Keynes-Ramsey rules.[10]

By transforming the FPEs from equations describing densities into equations describing distribution functions, we obtain a description of densities whose intuitive interpretation is very similar to derivations of less complex distributions as in Burdett and Mortensen (1998) or Burdett et al. (2011). In addition, however, our equations exhibit new "advection" terms that capture the shift of the distribution due to the evolution of the additional state variable, i.e. due to wealth.

The structure of the paper is as follows. Section 2 presents the model. Section 3 derives implications of optimal behaviour. Section 4 presents the phase diagram analysis to understand consumption-wealth patterns over time and across labour market states. Section 5 describes the joint distribution of the labour market status and wealth of one individual. The corresponding FPEs for constant relative risk aversion are derived, its properties and boundary conditions are discussed and an intuitive interpretation is provided. It also discusses how this approach can be used for other setups. It is also shown how constant absolute risk aversion changes the description of densities. Section 6 shows how to obtain the aggregate distribution of wealth and how to formulate appropriate initial distributions at the aggregate level. This allows to link macro to micro features of the model and to obtain a general equilibrium solution. The final section concludes.

2 The model

We consider a model where all aggregate variables are in a steady state. At the micro level, individuals face idiosyncratic uninsurable risk and variables evolve in a dynamic and stochastic way.

[10] We would like to thank Philipp Kircher for having put this so nicely.

2.1 Technologies

The production of output requires capital K and labour L. Both the capital stock and employment are endogenous but constant. The technology is given by $Y = Y(K, L)$ and $Y(.)$ has the usual neoclassical properties.

As is common for Mortensen-Pissarides type search and matching models, the employment status $z(t)$ of any individual jumps between the state of employment, w, and unemployment, b, with corresponding labour income w – the net wage – and unemployment benefits b. As an individual cannot lose her job when she does not have one and as finding a job makes (in the absence of on-the-job search) no sense for someone who has a job, both the job arrival rate $\mu(z(t))$ and the separation rate $s(z(t))$ are state dependent. As an example, when an individual is employed, $\mu(w) = 0$, when she is unemployed, $s(b) = 0$.

$z(t)$	w	b
$\mu(z(t))$	0	$\mu > 0$
$s(z(t))$	$s > 0$	0

Table 1 *State dependent arrival rates*

The process $z(t)$ is a continuous-time Markov chain with state space $\{w, b\}$. Intuitively, it can be described by the following stochastic differential equation,

$$dz(t) = \Delta dq_\mu - \Delta dq_s, \qquad \Delta \equiv w - b. \tag{1}$$

The Poisson process q_s counts how often our individual moves from employment into unemployment. The arrival rate of this process is given by $s(z(t))$. The Poisson process related to job finding is denoted by q_μ with an arrival rate $\mu(z(t))$. It counts how often the individual finds a job.

When the individual is employed, $z(t) = w$, the employment equation (1) simplifies to $dw = -(w - b)\, dq_s$. Whenever the process q_s jumps, i.e. when the individual loses her job and $dq_s = 1$, the change in labour income is given by $-w + b$ and, given that the individual earns w before losing the job, earns $w - w + b = b$ afterwards. Similarly, when unemployed, the employment status follows $db = (w - b)\, dq_\mu$ and finding a job, i.e. $dq_\mu = 1$, means that labour income increases from b to w.

The presentation in (1) is most useful for all "practical purposes", i.e. for solving the maximization problem and for the first step required in the derivation of the Fokker-Planck equations. Formally, we are aware that a continuous-time Markov chain representation of $z(t)$ is much more stringent. In fact, our companion paper (Bayer and Wälde, 2011) on the existence and stability of a unique stationary distribution explicitly follows this more rigorous approach.[11]

[11] If one tries to answer existence issues for Fokker-Planck equations, the continuous-time Markov chain approach is the preferred approach as well.

2.2 Households and government

Each individual can save in an asset a (which is capital used by firms). Her budget constraint reads

$$da(t) = \{ra(t) + z(t) - c(t)\}\, dt. \tag{2}$$

Per unit of time dt wealth $a(t)$ increases (or decreases) if capital income $ra(t)$ plus labour income $z(t)$ is larger (or smaller) than consumption $c(t)$. Following (1), labour income $z(t)$ is given either by w or b. Dividing the budget constraint by dt and using $\dot{a}(t) \equiv da(t)/dt$ would yield a more standard expression, $\dot{a}(t) = ra(t) + z(t) - c(t)$. As $a(t)$ is not differentiable with respect to time at moments where individuals jump between employment and unemployment (or vice versa), we prefer the above representation. The latter is also more consistent with (1).

The objective function of the individual is a standard intertemporal utility function,

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\, d\tau, \tag{3}$$

where expectations need to be formed due to the uncertainty of labour income which in turn makes consumption $c(\tau)$ uncertain. The expectations operator is E_t and conditions on the current state in t. The planning horizon starts in t and is infinite. The time preference rate ρ is positive.

Even though most of our results should hold for general instantaneous utility functions with positive but decreasing first derivatives, we will work with a CRRA specification,

$$u(c(\tau)) = \left\{ \begin{array}{l} \frac{c(\tau)^{1-\sigma}-1}{1-\sigma} \\ \ln c(\tau) \end{array} \right\} \text{ for } \left\{ \begin{array}{l} \sigma \neq 1 \text{ and } \sigma > 0, \\ \sigma = 1. \end{array} \right. \tag{4}$$

When illustrating properties of the wealth distributions, we will also use a CARA specification,

$$u(c(\tau)) = -e^{-\gamma c(\tau)}, \quad \gamma > 0, \tag{5}$$

where γ is the measure of absolute risk aversion. All formal proofs will use the CRRA specification for a positive measure of relative risk aversion $\sigma \neq 1$.

There is a government who can tax the gross wage $w/(1-\xi)$ using a proportional tax ξ. Tax income from employed workers is used to finance unemployment benefits b. The tax adjusts such that a static government budget constraint

$$\xi \frac{w}{1-\xi} L = b[N - L] \tag{6}$$

is fulfilled at each point in time. The path of benefits b is determined by some political process which is exogenous to this model. This process makes sure that benefits are smaller than the net wage, $b < w$.

2.3 Endowment

The workforce of this economy has an exogenous and invariant size N. Individuals are initially endowed with wealth $a_i(t)$. This can be a fixed number or random (see sect. 5). The capital stock is defined as the sum over individual wealth holdings,

$$K \equiv \Sigma_{i=1}^{N} a_i(t). \tag{7}$$

Given our steady state setup, the aggregate capital stock K is endogenous but constant. Loosely speaking, there is a very large number of agents i such that all dynamics at the individual level wash out at the aggregate level. See our definition of an equilibrium below – especially (14) – for a precise formulation.

Given the job separation and matching setup, it is well-known that in a steady state, aggregate employment is an increasing function of the matching and a decreasing function of the separation rate,

$$L = \frac{\mu}{\mu + s} N. \tag{8}$$

3 Optimality conditions and equilibrium

3.1 Keynes-Ramsey rules

For our understanding of optimal consumption behaviour, it is useful to derive a Keynes-Ramsey rule. We extend the approach suggested by Wälde (1999) for the case of an uncertain interest rate to our case of uncertain labour income. We suppress the time argument for readability. Consumption $c(a_w, w)$ of an employed individual with current wealth a_w follows (see app. B.1)

$$\begin{aligned} -\frac{u''(c(a_w,w))}{u'(c(a_w,w))} dc(a_w,w) = {} & \left\{ r - \rho + s \left[\frac{u'(c(a_w,b))}{u'(c(a_w,w))} - 1 \right] \right\} dt \\ & - \frac{u''(c(a_w,w))}{u'(c(a_w,w))} \left[c(a_w,b) - c(a_w,w) \right] dq_s \end{aligned} \tag{9}$$

while her wealth evolves according to (2) with $z = w$, i.e.

$$da_w = \left[r a_w + w - c(a_w,w) \right] dt. \tag{10}$$

Analogously, solving for the optimal consumption of an unemployed individual with current wealth a_b yields

$$\begin{aligned} -\frac{u''(c(a_b,b))}{u'(c(a_b,b))} dc(a_b,b) = {} & \left\{ r - \rho - \mu \left[1 - \frac{u'(c(a_b,w))}{u'(c(a_b,b))} \right] \right\} dt \\ & - \frac{u''(c(a_b,b))}{u'(c(a_b,b))} \left[c(a_b,w) - c(a_b,b) \right] dq_\mu \end{aligned} \tag{11}$$

and her wealth follows

$$da_b = [r a_b + b - c(a_b,b)] dt. \tag{12}$$

Without uncertainty about future labor income, i.e. $s = \mu = dq_s = dq_\mu = 0$, the above Keynes-Ramsey rules reduce to the classical deterministic consumption rule, $-\frac{u''(c)}{u'(c)}\dot{c} = r - \rho$. The additional $s[.]$ term in (9) shows that consumption growth is faster under the risk of a job loss. Note that the expression $[u'(c(a_w, b))/u'(c(a_w, w)) - 1]$ is positive as consumption $c(a_w, b)$ of an unemployed worker is smaller than consumption of an employed worker $c(a_w, w)$ (see lem. 8 for a proof) and marginal utility is decreasing, $u'' < 0$. Similarly, the $\mu[.]$ term in (11) shows that consumption growth for unemployed workers is smaller.

As the additional term in (9) contains the ratio of marginal utility from consumption when unemployed relative to marginal utility when employed, this suggests that it stands for precautionary savings (Leland, 1968, Aiyagari, 1994, Huggett and Ospina, 2001). When marginal utility from consumption under unemployment is much higher than marginal utility from employment, individuals experience a high drop in consumption when becoming unemployed. If relative consumption shrinks as wealth rises, i.e. if $\frac{d}{da}\frac{c(a,w)}{c(a,b)} < 0$, reducing this gap and smoothing consumption is best achieved by fast capital accumulation. This fast capital accumulation would go hand in hand with fast consumption growth as visible in (9).

In the case of unemployment, the $\mu[.]$ term in (11) suggests that the possibility to find a new job induces unemployed individuals to increase their current consumption level. Relative to a situation in which unemployment is an absorbing state (once unemployed, always unemployed, i.e. $\mu = 0$), the prospect of a higher labor income in the future reduces the willingness to give up today's consumption. With higher consumption levels, wealth accumulation is lower and consumption growth is reduced.

The stochastic dq-terms in (9) and (11) (tautologically) represent the discrete jumps in the level of consumption whenever the employment status changes. We will understand more about these jumps after the phase-diagram analysis below.

3.2 Factor rewards

There is random matching with arrival rate μ of workers to markets characterized by an infinite supply of jobs. Once a market is found, there is perfect competition and agents are price takers as in Lucas and Prescott (1974) or Moen (1997). Firms rent capital on a spot market and choose an amount such that marginal productivity equals the rental rate. At the aggregate level, this fixes capital returns r and the gross wage $w/(1-\xi)$ at

$$r = \frac{\partial Y(K, L)}{\partial K}, \qquad \frac{w}{1-\xi} = \frac{\partial Y(K, L)}{\partial L}. \tag{13}$$

3.3 Equilibrium

Consider one individual with an initial level of wealth of $a(t)$ and an employment status $z(t)$. This individual faces an uncertain future labour income stream $z(\tau)$. One can ask what the distribution of wealth of this individual for some long-run stationary state is. Denote the corresponding density by $p(a)$. Employing a law of large numbers (see below for detailed definitions and analysis), we can use this definition to define general equilibrium. There is a deterministic macro level where all variables are constant. All uncertainty and all dynamics take place at the micro level. The average capital stock (for N approaching infinity) is given by the mean of the wealth distribution, given a density $p(a)$ of wealth,

$$\frac{K}{N} = \int ap(a)\,da. \tag{14}$$

This provides the link between the micro and macro level. We can now formulate

Definition 1 *A competitive stationary equilibrium is described by a constant aggregate capital stock K and employment level L, factor rewards w, r and the tax rate ξ, two functions $c(a,w)$ and $c(a,b)$ and a wealth density $p(a)$ such that*

1. *K satisfies (14) and L is given by (8),*

2. *given exogenous benefits b, the government budget constraint (6) and the first-order condition for labour in (13) jointly fix the tax rate ξ and wage rate w, the interest rate r satisfies the first-order condition for capital in (13),*

3. *the consumption functions $c(a,z)$ satisfy the reduced form (21) plus two boundary conditions of def. 2,*

4. *the density $p(a)$ is the stationary distribution described by Fokker-Planck equations joint with initial conditions in sect. 5 and 6.*

In addition to this macro equilibrium, the dynamics of each individual's wealth distribution $p(a,z,\tau)$ is described by the solution to the same Fokker-Planck equations given initial conditions in sect. 5.

4 Consumption and wealth dynamics

Given our aggregate steady state, this section will now characterize optimal consumption and wealth dynamics of individuals in our economy.

4.1 Consumption growth and the interest rate

We first focus on individuals in periods between jumps. The evolution of consumption is then given by the deterministic part, i.e. the dt-part, in (9) and (11). We then easily understand

Lemma 1 *Individual consumption rises if and only if current consumption relative to consumption in the other state is sufficiently high.*

For the employed worker, consumption rises if and only if $c(a_w,w)$ relative to $c(a_w,b)$ is sufficiently high,

$$\frac{dc(a_w,w)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_w,b))}{u'(c(a_w,w))} \geq 1 - \frac{r-\rho}{s} \Leftrightarrow \frac{c(a_w,w)}{c(a_w,b)} \geq 1/\psi, \tag{15}$$

where

$$\psi \equiv \left(1 - \frac{r-\rho}{s}\right)^{-1/\sigma}. \tag{16}$$

For the unemployed worker, consumption rises if and only if $c(a_b, b)$ *relative to* $c(a_b, w)$ *is sufficiently high,*

$$\frac{dc(a_b, b)}{dt} \geq 0 \Leftrightarrow \frac{u'(c(a_b, w))}{u'(c(a_b, b))} \geq 1 - \frac{r-\rho}{\mu} \Leftrightarrow \frac{c(a_b, b)}{c(a_b, w)} \geq \left(1 - \frac{r-\rho}{\mu}\right)^{1/\sigma}. \tag{17}$$

Proof. Rearranging (9) and (11) for $dq_s = dq_\mu = 0$ and taking (4) into account gives the results (see app. B.2). Note that in what follows ψ will be used only for r sufficiently small making sure that ψ is a real number. ∎

We can now establish our first main findings. As the conditions in lem. 1 show, consumption and wealth dynamics crucially depend on how high the interest rate is. We therefore subdivide our discussion into three parts with r lying in the three ranges given by $(0, \rho]$, $(\rho, \rho + \mu)$, $[\rho + \mu, \infty)$. For the proofs of propositions 1 to 3, we rely on one very weak

Assumption 1 *Relative consumption* $c(a, w)/c(a, b)$ *is continuously differentiable in wealth* a. *The number of sign changes of its first derivative with respect to wealth in any interval of finite length is finite.*[12]

Starting with the third range $[\rho + \mu, \infty)$, we obtain

Proposition 1 *For a high interest rate, i.e. if* $r \geq \rho + \mu$, *consumption of employed and unemployed workers always increases.*

Proof. Consumption of the employed worker increases as can be directly seen from the first expression in (15). As long as $r > \rho$ and $c(a, w) > c(a, b)$, the latter is proven in lem. 8, condition (15) is fulfilled: The right-hand side (RHS) is smaller than one and the left-hand side is larger than one as long as $u'' < 0$ which holds for (4). The case of the unemployed worker can also most easily be seen from the first expression in (17). For $r = \rho + \mu + \varepsilon$ with $\varepsilon \geq 0$, the RHS is given by $1 - \frac{r-\rho}{\mu} = -\frac{\varepsilon}{\mu} \leq 0$. As $\frac{u'(c(a_b, w))}{u'(c(a_b, b))} \geq 0$, (17) holds for $r \geq \rho + \mu$. ∎

The high interest rate case reminds of the standard optimal saving result in deterministic setups. If the interest rate is only high enough, consumption and wealth increase over time. This is true here as well. The only difference consists in the fact that the interest rate must be higher than the time preference rate *plus* the job arrival rate.

While we leave a quantitative analysis to ongoing numerical work, it is interesting already at this stage to note that the difference for the interest rate as compared to

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[12] The second sentence of this assumption is required to rule out "pathological cases". One can construct continuously differentiable functions that change sign infinitly often in a finite neighborhood (think of $x \sin(1/x)$ in a neighborhood of zero). None of these functions would be economically plausible in any way. We employ this assumption neither for our other proofs in this nor for the proofs in the companion paper.

deterministic models is quite substantial. In deterministic models, the interest rate must be larger than the time preference rate. As the job arrival rate is around four times higher than the time preference rate, the interest rate must be much higher here to guarantee wealth growth in all employment states.

As in other setups with growing consumption, we need to make sure that consumption does not grow too fast. If it does, utility grows too fast and the expected value of the integral in the objective function (3) is not finite. Optimization would then be more involved, which we would like to avoid. We therefore have to impose a boundedness condition which implies an upper limit on the interest rate. This condition can easily be derived for the limiting case where a is very large, i.e. where the difference between w and b can be neglected. The boundedness condition then reads $(1-\sigma)\,r < \rho$.[13]

The second result is summarized in

Proposition 2 *If the interest rate is at an intermediate level, i.e. $\rho < r < \rho + \mu$,*

(i) consumption of employed workers always increases.

(ii) consumption of an unemployed worker increases only if she is sufficiently wealthy, i.e. if her wealth a exceeds the threshold level a_b^*, where the threshold level is implicitly given by

$$\frac{u'\left(c\left(a_b^*, w\right)\right)}{u'\left(c\left(a_b^*, b\right)\right)} \equiv 1 - \frac{r-\rho}{\mu}. \tag{18}$$

Consumption decreases for $a < a_b^*$.

(iii) At the threshold level a_b^*, consumption of employed workers exceeds consumption of unemployed workers.

Proof. The proof is in complete analogy to the proof of the following prop. 3 for the low interest rate. As prop. 3 is more important for our purposes, we will prove prop. 3 but not this one. ■

This proposition points to the central new insight for optimal consumption. For the employed worker, the result from deterministic worlds survives: If the interest rate is higher than the time preference rate, consumption and wealth rise. For the unemployed worker, however, this is not true. Consumption and wealth rise only if the unemployed worker is sufficiently rich. In a way, this is a "dramatic" result. If a worker loses a job, consumption continues to rise only if the worker is sufficiently rich at the moment of the job loss. If, by contrast, a worker losing a job is below the threshold level a_b^*, consumption and wealth is reduced.

Finally, we have

Proposition 3 *Consider a low interest rate, i.e. $0 < r \leq \rho$. Define a threshold level a_w^* by*

$$\frac{u'\left(c\left(a_w^*, b\right)\right)}{u'\left(c\left(a_w^*, w\right)\right)} \equiv 1 - \frac{r-\rho}{s}. \tag{19}$$

[13] An interest rate r can satisfy both this boundedness condition and the condition $r \geq \rho + \mu$ for the high-interest-rate case if $\mu < \frac{\sigma}{1-\sigma}\rho$. This condition on μ needs to be taken into account in any quantitative analysis.

For our instantaneous utility function (4), this definition reads

$$c(a_w^*, b) = \psi c(a_w^*, w) \tag{20}$$

where ψ is from (16).

(i) Consumption of employed workers increases if the worker owns a sufficiently low wealth level, $a < a_w^*$. Employed workers with $a > a_w^*$ choose falling consumption paths.

(ii) Consumption of unemployed workers always decreases.

(iii) Consumption of employed workers exceeds consumption of unemployed workers at the threshold a_w^*, i.e. $\psi \leq 1$ in (20) for $r \leq \rho$.

Proof. see app. A.1 ■

We are now in a position to intuitively understand all three propositions. In deterministic setups, an interest rate exceeding the time preference rate is enough to imply positive consumption growth. In a world with precautionary saving, only employed workers will experience rising consumption for sure when $r > \rho$. Unemployed workers experience rising consumption only for a high interest rate $r > \rho + \mu$ or for r close to but larger than ρ only if they are sufficiently rich. The reason for these results is the "optimism" of unemployed workers that they will find a job in the future. Anticipating higher future income, they choose a higher consumption level than in a situation where the state of unemployment is permanent. Due to this higher consumption level, consumption and wealth growth is reduced. Only if the interest rate exceeds $\rho + \mu$ or if an unemployed worker is sufficiently rich, this higher consumption does still allow for consumption growth.

Similarly for employed workers: In deterministic worlds, an interest rate below the time preference rate implies falling consumption and wealth levels. Here, as there is precautionary saving of the employed worker, a situation of $r < \rho$ still implies growing consumption and wealth.

These propositions also clearly show that if we are interested in a general equilibrium result with stationary properties, the interest rate cannot be larger than the time preference rate. If the interest rate exceeded the time preference rate, consumption would grow without bound – at least for some employment states and levels of wealth. Only for $r \leq \rho$ there are consumption dynamics which indicate that a stationary distribution of consumption can exist.

4.2 The reduced form

Before we can derive further properties of optimal behaviour, we need a "reduced form" for optimal behaviour of individuals. A reduced form is a system of equations with as few equations as possible which determines an identical number of endogenous variables and which allow us to derive all other endogenous variables subsequently. When searching for such a reduced form, we can exploit the fact that Poisson uncertainty allows to divide the analysis of a system into what happens between jumps and what happens at jumps. Between jumps, the system evolves in a deterministic way –

but does of course take the possibility of a jump into account as is clearly visible in the precautionary savings terms in the Keynes-Ramsey rules (9) and (11).[14]

We obtain such a reduced form by focusing on the evolution between jumps and by eliminating time as exogenous variable. Computing the derivatives of consumption with respect to wealth in both states and considering wealth as the exogenous variable, we obtain a two-dimensional system of non-autonomous ordinary differential equations (ODE). As wealth is now the argument for these two differential equations, there is no longer a need to distinguish between wealth of employed and unemployed workers (i.e. between a_w and a_b). We simply ask how wealth changes in one or the other state given a certain wealth level a. Between jumps, the reduced form therefore reads

$$-\frac{u''(c(a,w))}{u'(c(a,w))}\frac{dc(a,w)}{da} = \frac{r-\rho+s\left[\frac{u'(c(a,b))}{u'(c(a,w))}-1\right]}{ra+w-c(a,w)}, \tag{21a}$$

$$-\frac{u''(c(a,b))}{u'(c(a,b))}\frac{dc(a,b)}{da} = \frac{r-\rho-\mu\left[1-\frac{u'(c(a,w))}{u'(c(a,b))}\right]}{ra+b-c(a,b)}. \tag{21b}$$

With two boundary conditions, this system provides a unique solution for $c(a,w)$ and $c(a,b)$. Once solved, the effect of a jump is then simply the effect of a jump of consumption from, say, $c(a,w)$ to $c(a,b)$.

4.3 Phase diagram and policy functions

Given the findings on consumption in the above propositions and our reduced form in (21), we can now describe the link between optimal consumption and wealth of unemployed and employed workers. We will focus on the case of an interest rate below the time preference rate as this is implies a stationary general equilibrium solution. We leave general equilibrium analyses of the other cases for future work.

- Natural borrowing limit

The subsequent analysis will be facilitated by noting that there is an endogenous "natural" borrowing limit. The idea is similar to Aiyagari's (1994) borrowing limit resulting from non-negative consumption. This limit is derived in the following

Proposition 4 *Any individual with initial wealth $a \geq -b/r$ will never be able to or willing to borrow more than $-b/r$. Consumption of an unemployed worker at $a = -b/r$ is zero, $c(-b/r, b) = 0$.*

[14] One could be tempted to think of the deterministic parts of the two Keynes-Ramsey rules (9) and (11), jointly with the budget constraints (10) and (12) to provide such a reduced form. With an initial condition for wealth and the consumption levels in the different states, one could think of the evolution between jumps as being described by four ordinary differential equations. When solving these equations (conceptionally or numerically), the solution in t for consumption of, say, the unemployed, $c(a_b, b)$ from (11) would not correspond to consumption $c(a_w, b)$ as required in the precautionary savings part in (9) for the employed as wealth levels are accumulated at different speed, i.e. $a_b(t)$ generally differs from $a_w(t)$. Equations (9) to (12) do therefore not constitute a system of ODEs and cannot be used as a reduced form.

Proof. "willing to": An employed individual with $a \geq -b/r$ will increase wealth for any wealth levels below a_w^* from (19). If a_w^* is larger than $-b/r$ – which we can safely assume – employed workers with wealth below a_w^* increase wealth and are not willing to borrow more than $-b/r$.

"able to": Imagine an unemployed worker had wealth lower than $-b/r$. Even if consumption is equal to zero, wealth would further fall, given that $\dot{a} = ra + b < 0 \Leftrightarrow a < -b/r$. If an individual could commit to zero consumption when employed and if the separation rate was zero, the maximum debt an individual could pay back is $-w/r$. Imagine an unemployed worker succeeded in convincing someone to lend her "money" even though current wealth is below $-b/r$. Then, with a strictly positive probability, wealth will fall below $-w/r$ within a finite period of time. Hence, anyone lending to an unemployed worker with wealth below $-b/r$ knows that not all of this loan will be paid back with positive probability. This cannot be the case in our setup with one riskless asset. Hence, the maximum debt level is b/r and consumption is zero at $a = -b/r$ for an unemployed worker. ■

- Laws of motion and policy functions

The following fig. 1 plots wealth on the horizontal and consumption $c(a, z)$ on the vertical axis. It plots dashed zero-motion lines for a_w and $c(a, w)$ and a solid zero-motion line for a_b following from (10), (19) and (12), respectively. We assume for this figure that the threshold level a_w^* is positive.[15] The intersection point of the zero-motion lines for $c(a, w)$ and a_w is the temporary steady state (TSS),

$$\Theta \equiv (a_w^*, c(a_w^*, w)) . \tag{22}$$

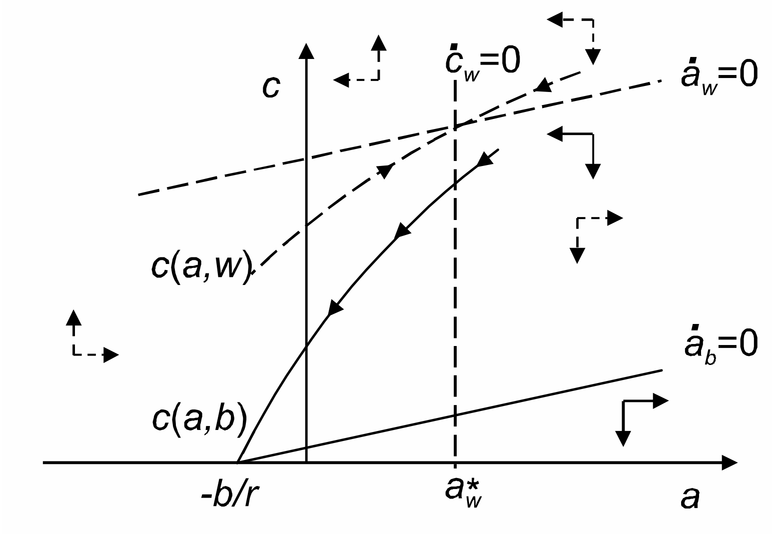

Figure 1 *Policy functions for employed and unemployed workers (low interest rate)*

[15]This is of course a quantitative issue. In ongoing numerical work, the threshold is positive for reasonable parameter values. It approaches infinity for r approaching ρ.

We call this point *temporary* steady state for two reasons. On the one hand, employed workers experience no change in wealth, consumption or any other variable when at this point (as in a standard steady state of a deterministic system). On the other hand, the expected spell in employment is finite and a random transition into unemployment will eventually occur. Hence, the state in Θ is steady only temporarily.

As we know from prop. 3 that consumption for the unemployed always falls, both consumption and wealth fall above the zero-motion line for a_b. The arrow-pairs for the employed workers are also added. They show that one can draw a saddle-path through the TSS. To the left of the TSS, wealth and consumption of employed workers rise, to the right, they fall.

Relative consumption when the employed worker is in the TSS is given by (20). A trajectory going through $(a_w^*, c(a_w^*, b))$ and hitting the zero-motion line of a_b at $-b/r$ is in accordance with laws of motions for the unemployed worker.

- Properties of optimal behaviour

The case of a low interest rate is particularly useful as the range of wealth a worker can hold is bounded. Whatever the initial wealth level, there is a positive probability that the wealth level will be in the range $[-b/r, a_w^*]$ after some finite length of time. For an illustration, consider the policy functions in fig. 1: Wealth decreases both for employed and unemployed workers for $a > a_w^*$. The transition into the range $[-b/r, a_w^*]$ will take place only in the state of unemployment which, however, occurs with positive probability.

When wealth of an individual is within the range $[-b/r, a_w^*]$, consumption and wealth will rise while employed and fall while unemployed. While employed, precautionary saving motives drive the worker to accumulate wealth. While unemployed, the worker runs down current wealth as higher income for the future is anticipated – "postcautionary dissaving" takes place. When a worker loses a job at a wealth level of, say, $a_w^*/2$, his consumption level will drop from $c(a_w^*/2, w)$ to $c(a_w^*/2, b)$. Conversely, if an unemployed worker finds a job at, say, $a = 0$, her consumption increases from $c(0, b)$ to $c(0, w)$. A worker will therefore be in a permanent consumption and wealth cycle. Given these dynamics, one can easily imagine a distribution of wealth over the range $[-b/r, a_w^*]$.

4.4 Existence of an optimal consumption path

All steps undertaken so far were explorative. We now turn to a proof for the existence of a path $c(a, z)$ as depicted in fig. 1.

In fig. 1, we implicitly considered solutions of our system in the set $Q = \{a \geq -b/r\} \cap \{c(a, w) \leq ra + w\} \cap \{c(a, b) \geq ra + b\} \cap \{c(a, b) \geq 0\} \cap \{c(a, w) \geq c(a, b)\}$. In words, wealth is at least as large as the maximum debt level b/r, consumption of the employed worker is below the zero-motion line for her wealth, consumption of the unemployed worker is above her zero-motion line for wealth, consumption of the unemployed worker is non-negative and consumption of employed workers always exceeds consumption of unemployed workers (see lem. 8).

For the proofs we restrict this set in two ways. First, we consider the domain

$$Q_v = \{(a, c(a, w), c(a, b)) \in \mathbb{R}^3 | (a, c(a, w), c(a, b)) \in Q, c(a, w) \leq ra + w - v\}, \quad (23)$$

where v is the small positive constant, as an approximation to our "full" set Q. As $Q_0 = Q$, Q_v simply excludes the zero-motion line for wealth of the employed workers. We need to do this as the fraction on the right-hand side of our differential equation (21a) is not defined for the TSS.[16] As v is small, however, we can get arbitrarily close to this zero-motion line and Q_v approximates Q arbitrarily well.

Second, we consider

$$\begin{aligned} R_{v,\Psi} = \big\{ &(a, c(a, w), c(a, b)) \in \mathbb{R}^3 | (a, c(a, w), c(a, b)) \in Q_v, \\ &c(a, w) \leq \Psi < \infty, \ a \leq (c(a, w) - w + v)/r \big\}, \end{aligned} \quad (24)$$

where Ψ is a finite large constant.[17] This additional restriction makes the set $R_{v,\Psi}$ bounded. This is a purely technical necessity.

We now introduce an auxiliary TSS (aTSS) in order to capture v. In analogy to the TSS Θ from (22), this point is defined by

$$\Theta_v \equiv (a_w^*, c_v(a_w^*, w)),$$

i.e. the wealth level a_w^* is unchanged but the consumption level is "a bit lower" than in the TSS. In the TSS, the consumption level is on the zero-motion line, i.e. $c(a_w^*, w) = ra_w^* + w$. In the aTSS, the consumption level is on the line $ra + w - v$ and therefore given by $c_v(a_w^*, w) = ra_w^* + w - v$. Let us now consider the following

Definition 2 *(Optimal consumption path) A* consumption path *is a solution* $(a, c(a, w), c(a, b))$ *of the ODE-system (21) for the range* $-b/r \leq a \leq a_w^*$ *in* $R_{v,\Psi}$ *with terminal condition* $(a_w^*, c_v(a_w^*, w), c_v(a_w^*, b))$. *In analogy to the aTSS and to (20), the terminal condition satisfies* $c_v(a_w^*, w) = ra_w^* + w - v$ *and* $c_v(a_w^*, b) = \psi c_v(a_w^*, w)$ *for an arbitrary* $a_w^* > -b/r$. *An* optimal consumption path *is a consumption path which in addition satisfies* $c(-b/r, b) = 0$.

App. A.2 then proves

Theorem 1 *There is an optimal consumption path.*

This establishes that we can continue in our analysis by taking the existence of a path $c(a, z)$ as given. Intuitively speaking, i.e. looking at v as very small constant close to zero, we know that there are paths $c(a, w)$ and $c(a, b)$ as drawn in fig. 1. The approximation implied by the auxiliary TSS is very small compared to any measurement error in the data. Values of $v = 10^{-3}$ worked perfectly in numerical solutions.

[16] While this is a standard property of many steady states, the standard solutions (e.g. linearization around the steady state) do not work in our case. This is in part due to the fact that the original stochastic differential equation system (9) to (12) - even when stripped of its stochastic part - is not an ordinary differential equation system.

[17] The constant Ψ only serves to make $R_{v,\Psi} \subset \mathbb{R}^3$ a compact set, which we need to obtain global, uniform Lipschitz constants. We shall see below that Ψ has to be chosen larger than $\Psi_0 = \frac{\psi w - b}{(1-\psi)r}$. In this case, however, Ψ does not interfere with the construction.

5 The distribution of labour income and wealth

We now come to the main contribution of this paper where we describe distributional properties of $z(t)$ and $a(t)$. This is of importance per se from a micro perspective – but it will also allow us to close the model and obtain general equilibrium results. As argued in the introduction, the basic structure we will get to know is a structure that serves as an example that can be adapted for many other applications.

5.1 Labour market probabilities

Consider first the distribution of the labour market state. Given that the transition rates between w and b are constant, the conditional probabilities of being in state $z(\tau)$ follow e.g. from solving Kolmogorov's backward equations as presented e.g. in Ross (1993, ch. 6). As an example, the probability of being employed in $\tau \geq t$ conditional on being in state $z \in \{w, b\}$ in t are

$$P(z(\tau) = w | z(t) = w) \equiv p_{ww}(\tau) = \frac{\mu}{\mu + s} + \frac{s}{\mu + s} e^{-(\mu+s)(\tau-t)}, \tag{25}$$

$$P(z(\tau) = w | z(t) = b) \equiv p_{bw}(\tau) = \frac{\mu}{\mu + s} - \frac{\mu}{\mu + s} e^{-(\mu+s)(\tau-t)}. \tag{26}$$

The complementary probabilities are $p_{wb}(\tau) = 1 - p_{ww}(\tau)$ and $p_{bb}(\tau) = 1 - p_{bw}(\tau)$. Letting $p_w(t)$ denote the probability of $z(t) = w$, i.e. letting it describe the initial distribution of $z(t)$, the unconditional probability of being in state z in τ is

$$p_z(\tau) = p_w(t) p_{wz}(\tau) + (1 - p_w(t)) p_{bz}(\tau). \tag{27}$$

Equations (25) and (26) nicely show the influence of the initial condition on the probability of having a job. Consider a point in time τ which is just an instant after t. Let this instant be so small that τ is basically identical to t. Then, the probability of being employed in τ (where $\tau = t$) is given by $\frac{\mu}{\mu+s} + \frac{s}{\mu+s} = 1$. Similarly, the probability of being unemployed in τ where τ is very close to t is given by (set $\tau = t$ in (26)) $\frac{\mu}{\mu+s} - \frac{\mu}{\mu+s} = 0$. The longer the point τ lies into the future, the less important the initial state becomes and the closer both probabilities approach the unconditional probability of being employed, which is $\frac{\mu}{\mu+s}$.

5.2 Fokker-Planck equations for wealth

5.2.1 The question and how to answer it

Now consider one individual with a level of wealth of $a(t)$ and an employment status $z(t)$. This individual faces an uncertain future labour income stream $z(\tau)$. Our fundamental question is: what is the joint distribution of $a(\tau)$ and $z(\tau)$ for $\tau \geq t$?

In order to answer this question, or to answer any question of this type, we need a description of the stochastic processes of $a(\tau)$ and $z(\tau)$. The process for $z(\tau)$ is given in (1). The process for $a(\tau)$ is given by the budget constraint (2), where, however, consumption needs to be replaced by optimal consumption $c(a, z)$. Then, after defining the (joint) density of $(a(\tau), z(\tau))$, i.e. of the labour market status and wealth for $\tau \geq t$, we can apply the "Fokker-Planck machinery" to obtain a description of the densities.

We denote the joint density by $p(a, z, \tau)$. For each point in time τ, there is obviously a discrete and a continuous random variable. We can therefore split the density into two "subdensities" $p(a, w, \tau)$ and $p(a, b, \tau)$, both drawn in fig. 2 for some $\tau \geq t$. The subdensities can be understood as the product of a conditional density $p(a, \tau|z)$ times the probability of being in employment state z,

$$p(a, z, \tau) \equiv p(a, \tau|z) \, p_z(\tau). \tag{28}$$

The probability $p_z(\tau)$ of an individual to be in a state z in τ is given by (27). As is clear from (28), $p(a, z, \tau)$ are not conditional densities – they rather integrate to the probability of $z(\tau) = z$. Looking at an individual who is in state z in τ, we get

$$\int p(a, z, \tau) \, da = \int p(a, \tau|z) \, p_z(\tau) \, da = p_z(\tau) \int p(a, \tau|z) \, da = p_z(\tau). \tag{29}$$

The density of a at some point in time τ is then simply

$$p(a, \tau) = p(a, w, \tau) + p(a, b, \tau). \tag{30}$$

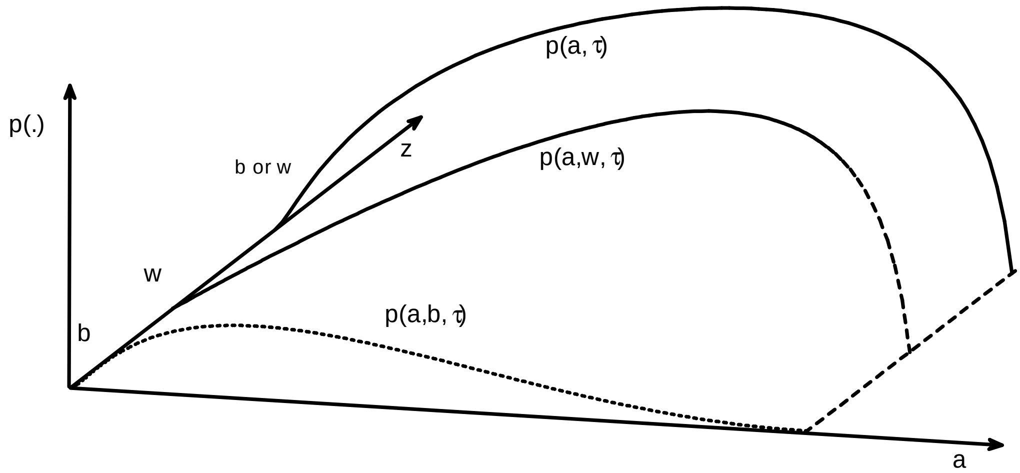

Figure 2 *The subdensities $p(a, b, \tau)$ and $p(a, w, \tau)$ and the density $p(a, \tau)$*

Note that the distribution of $(a(\tau), z(\tau))$ certainly depends on the initial condition $(a(t), z(t))$, which needs to be specified in order to calculate $p(a, z, \tau)$. In the notation we do not distinguish between the following two possibilities. Firstly, $(a(t), z(t))$ can be deterministic numbers, in which case $p(a, z, t)$ is a Dirac-distribution centered in $(a(t), z(t))$ (more precisely, the mapping $a \rightarrow p(z, a, t)$ is a Dirac-distribution). Secondly, $(a(t), z(t))$ can itself be random, either because we regard them as outcomes of the employment-wealth-process started at an even earlier time, or because there is some intrinsic uncertainty in measuring $a(t)$ (see below in sect. 5.2.3).

Let us now step back and ask how this approach can be applied to other setups. If one would like to understand the process of accumulation and depreciation of skills and experience during different employment states, one would have to specify a differential equation for skill similar to the budget constraint (2). Joint with the fundamental process (1) one could then derive Fokker-Planck equations for densities. If one would like to model the endogenous distribution of entitlement to unemployment benefits, one would have to "translate" regulations concerning entitlement into a differential equation, add again (1) and proceed to derive Fokker-Planck equations. Similar procedures are possible for analysing distributions over the business cycle where some aggregate shock process would be added to (2), (1) or both. Note that this approach works for processes driven e.g. by Brownian motion just as well.

5.2.2 The equations and their economic interpretation

The derivation of the Fokker-Planck equations is in app. A.3. The result is a system of two non-autonomous quasi-linear partial differential equations in $p(a, w, \tau)$ and $p(a, b, \tau)$,

$$\frac{\partial}{\partial \tau} p(a, w, \tau) + \{ra + w - c(a, w)\} \frac{\partial}{\partial a} p(a, w, \tau) = \\ -\left\{r - \frac{\partial}{\partial a} c(a, w) + s\right\} p(a, w, \tau) + \mu p(a, b, \tau), \tag{31a}$$

$$\frac{\partial}{\partial \tau} p(a, b, \tau) + \{ra + b - c(a, b)\} \frac{\partial}{\partial a} p(a, b, \tau) = \\ sp(a, w, \tau) - \left\{r - \frac{\partial}{\partial a} c(a, b) + \mu\right\} p(a, b, \tau). \tag{31b}$$

The system is a *partial* differential equation system as there are two derivatives, one with respect to time τ and one with respect to wealth a – which is not surprising: As the FPEs describe the evolution of the density for wealth over time, two derivatives are needed. The derivative with respect to a describes the "cross-sectional" property of the density for a given τ. The time derivative describes how a density changes over time.[18] The differential equations are called *quasi*-linear as the factors in front of the wealth-derivatives are functions of a. When we analyse the distribution of wealth for CARA utility (5), we obtain a *linear* PDE system (see sect. 5.2.6). The PDEs are *non-autonomous* as some of the terms (other than the densities) also depend explicitly on one of the exogenous variables (exogenous in a differential equation sense), i.e. on wealth a.

As we can see, the density depends on properties of optimizing behaviour through the consumption levels $c(a, w)$ and $c(a, b)$ and through the marginal propensities to

[18] Compare this to the Pearson system of distributions that describes densities by ordinary non-autonomous differential equations (see e.g. Johnson, Kotz and Balakrishnan, 1994, ch. 12). These ordinary differential equations describe the densisty of one random variable. Here, we analyse a stochastic process, i.e. a sequence of random variables, and therefore need two derivatives.

consume out of wealth, $\partial c(a, w)/\partial a$. These FPEs therefore describe the evolution of wealth for any specification of the utility function (e.g. CRRA, CARA, log, etc.). Modifying the utility function (e.g. allowing for labour supply or separating the intertemporal elasticity of substitution from risk aversion) affects the density of wealth through the effect on the optimal consumption plan $c(a, z)$.

Before we given an economic interpretation to these equations, we transform them such that they do not describe densities but distribution functions. To this end, define subdistribution functions as

$$P(a, z, \tau) \equiv \int_{-b/r}^{a} p(a, z, \tau)\, da. \tag{32}$$

The term $P(a, w, \tau)$ gives the probability that an individual will be employed in τ and own wealth equal or lower to a. Given our definition of subdensities and their property in (29), we know that $\lim_{a\to\infty} P(a, w, \tau) = p_{zw}(\tau)$ where the term $p_{zw}(\tau)$ is given in either (25) or (26), depending on the initial state in t.

The transformation of our FPEs is subject to the condition that $p\left(-\frac{b}{r}, z, \tau\right) = 0$ for all τ. This means that there is no worker with wealth equal to $-b/r$. As a wealth of $-b/r$ for unemployed workers would imply zero consumption, $c(-b/r, b) = 0$, this can be ruled out indeed as marginal utility from consumption would then be infinity. This would violate optimality. As employed workers with wealth of $-b/r$ can only originate from unemployed workers with this wealth level (as wealth of employed workers increases, see fig. 1) and as $p\left(-\frac{b}{r}, b, \tau\right) = 0$ for all τ, we know that $p\left(-\frac{b}{r}, w, \tau\right) = 0$ for all τ as well.

The subdistribution functions in (32) obey the following system (cf. app. B.4)

$$\frac{\partial}{\partial \tau} P(a, w, \tau) = -\{ra + w - c(a, w)\} \frac{\partial}{\partial a} P(a, w, \tau) - sP(a, w, \tau) + \mu P(a, b, \tau), \tag{33a}$$

$$\frac{\partial}{\partial \tau} P(a, b, \tau) = -\{ra + b - c(a, b)\} \frac{\partial}{\partial a} P(a, b, \tau) + sP(a, w, \tau) - \mu P(a, b, \tau). \tag{33b}$$

This system is now extremely easy to understand: Starting with the first equation, the evolution of the distribution function over time, i.e. the time derivative $\partial P(a, w, \tau)/\partial \tau$ on the left hand sides depends on three terms. Starting at the end, there is an increase in the probability $P(a, w, \tau)$ if there is a high flow from the state of being unemployed. This flow can be high if the matching rate μ, the probability of being unemployed $P(a, b, \tau)$ or if a combination of the two is high. Similarly, the probability $P(a, w, \tau)$ decreases (ceteris paribus) exponentially at the rate s, and the faster so, the higher the separation rate. The interpretation of the last two terms in the second equation (33b) is identical (subject to reversed signs). These two terms are very familiar from derivations of wage distributions in the Burdett-Mortensen (1998) tradition.

The first term is what is called an advection term in physics and related disciplines where the flow of particles in a fluid is being modelled. Particles move stochastically within the fluid and are subject to advection, i.e. to movements due to the movement of the fluid itself ("bulk motion"). In this economic application, wealth of workers is flowing in an economy. The wealth levels of workers are stochastically moving back and forth between different states w and b. These are the two terms at the end of (33a,b). Wealth is also moved non-stochastically within the states, either upwards (when employed) or downwards (when unemployed). The direction of the movement is on the wealth line, i.e. the partial derivative $\partial P(a, w, \tau) / \partial a$ is denotes the direction of a.[19] The speed of this movement is determined by savings $ra + z - c(a, z)$. The speed is positive when employed and negative when unemployed. The overall effect of positive savings for the probability $P(a, w, \tau)$ of employed workers is then to *decrease* this probability. As wealth increases, the probability of having a wealth level equal to or lower than a certain level a obviously falls as there is a permanent flow towards higher wealth levels. This flow is then reversed in the state of unemployment where the speed (i.e. savings $ra - b - c(a, b)$) is negative. As a consequence, the probability $P(a, b, \tau)$ ceteris paribus increases over time as unemployed workers "gather" towards the lower end of the wealth distribution.

5.2.3 Initial conditions

Obtaining a unique solution for ODEs generally requires certain differentiability conditions and as many initial conditions as differential equations. Conditions for obtaining a unique solution for PDEs differ in various respects, of which the most important one from an intuitive perspective is the fact that instead of initial conditions (i.e. an initial value or vector), initial functions are required. This can easily be understood for our case: Let us assume two initial functions for a, one for each labour market state $z \in \{w, b\}$. The obvious interpretation for these initial functions are densities, just as illustrated in fig. 2. Initial functions would therefore be given by $p(a, b, t) = p^{ini}(a, b)$ and $p(a, w, t) = p^{ini}(a, w)$. Clearly, they take positive values on the range $[-b/r, a_w^*]$ only and need to jointly integrate to unity. Given these initial functions, one can then compute the partial derivatives with respect to a in (31). This gives an ODE system which allows us to compute the density for the "next" τ. Repeating this gives us the densities for all z, a and τ we are interested in.

An initial function for wealth in each labour market state sounds unusual when thinking of one individual who, say, in t has wealth of $a(t)$ and is currently employed, $z(t) = w$. One can express these two deterministic numbers such that we obtain initial functions, however. First, $p^{ini}(a, b) = 0$: as the probability for an employed individual to be unemployed is zero and the probability of being unemployed is given by $\int_{-b/r}^{a_w^*} p^{ini}(a, b)\, da$ (compare the example in (29)), $p^{ini}(a, b)$ must be zero. Second, there are two possibilities for $p^{ini}(a, w)$. Either one considers $p^{ini}(a, w)$ as a Dirac-distribution, i.e. there is a degenerate density with mass-point at $a = a(t)$. Or, maybe most convenient both for numerical purposes and for intuition, one considers the current wealth level $a(t)$ to be observed with some imprecision. Pricing various

[19] Particles in a fluid can move in three dimensions, left-right, up-down and back-forth. The advection term would then have three partial deriviatives, one in each direction.

types of assets (cars or other durable consumption goods like a house) might not be straightforward and one can easily imagine an initial function which is zero to the left of $a_{\min}$ and to the right of $a_{\max}$ and condenses all probability between these values (which can of course be arbitrarily close to $a(t)$).[20]

5.2.4 A density gives a density

The Fokker-Planck equations have a very convenient property that easily allows to show that they indeed describe densities (in the sense that their solutions integrate to one). The only condition is that the initial functions integrate to one. We summarize this in the following

Proposition 5 *Define $I(\tau) \equiv \int_{-\infty}^{\infty} p(a,w,\tau)+p(a,b,\tau)\,da$. Given the laws of motion for $p(a,z,\tau)$ from (31) and the fact of a bounded support $[-b/r, a_w^*]$, this integral is mass-preserving, i.e. $dI(\tau)/d\tau = 0$ for all τ. Assuming initial densities, i.e. initial functions $p(a,z,t) \geq 0$ such that $I(t) = 1$, the PDEs in (31) indeed describe the dynamics of distributions over time.*

Proof. see app. B.3 ■

This is an extremely useful property as this implies that with an initial density we know that all other functions $p(a,w,\tau)+p(a,b,\tau)$ integrate to one and therefore represent densities.

5.2.5 The long-run distribution of individual wealth

When we are interested in the long-run distribution of wealth and income only, the time derivatives of the densities would be zero and the long-run densities would be described by two linear ordinary differential equations. This is true both for the system in densities (31) and for the system for distributions (33). Both of these systems can be solved numerically with standard packages.

The advantage of the FPEs for densities (31) consists in the fact that boundary conditions are provided by the analysis of optimal consumption, see e.g. fig. 1. These boundary conditions are

$$p(a_w^*, w) = 0, \quad p(a_w^*, b) = 0. \tag{34}$$

The intuition for $p(a_w^*, w) = 0$ comes from the saddle-path nature of the TSS Θ in (22): There is one path going into Θ from the left and one going into Θ from the right and two (not drawn) starting from Θ and going North and South. In saddle-points of ODE systems, one can prove by linearization around the fix point that local solutions of the ODE approach the saddle point asymptotically. Linearization here is more involved given the special structure of our system (see fn. 16). Assuming that the qualitative properties of local behaviour are not affected by this structure, we would observe asymptotic behaviour here as well and the TSS Θ would actually

[20] Our companion paper (Bayer and Wälde, 2011) proves that all initial distributions within the range $[-b/r, a_w^*]$ converge to a unique stable distribution in the long run.

never be reached: $p(a_w^*, w) = 0$ would follow. The second boundary condition is then an immediate consequence. As the state (a_w^*, b) can occur only through a transition from (a_w^*, w) but the density at (a_w^*, w) is zero, $p(a_w^*, b) = 0$ as well.

5.2.6 The CARA case

For many purposes it is highly useful to work with a CARA utility function (see e.g. Shimer and Werning, 2007, 2008 for models that include capital accumulation). If we assume a CARA utility function as in (5), the FPEs (31) simplify dramatically and become an autonomous *linear* partial differential equation system,

$$\frac{\partial}{\partial t}p(a,w,t) = m_w \frac{\partial}{\partial a}p(a,w,t) - sp(a,w,t) + \mu p(a,b,t),$$
$$\frac{\partial}{\partial t}p(a,b,t) = m_b \frac{\partial}{\partial a}p(a,b,t) + sp(a,w,t) - \mu p(a,b,t).$$

Letting the optimal consumption path under CARA be given by $c(a,z) = ra + z + m_z$, the parameters m_z need to be such that Bellman equations for the states of employment and unemployment are satisfied.

Any densities implied by models of this type must obey these equations. It is known that there are various closed-form solutions for these equations for specific initial distributions of wealth. It is an open question whether these closed-form solutions actually describe densities.[21] The method of characteristics is the standard tool to transform PDE systems into (larger) ODE systems and to understand the dynamics of distributions analytically or numerically. We leave this for future research.

6 The aggregate distribution of wealth and employment

Using all the results we collected so far on individual behaviour, we are now in an easy position to describe the aggregate distribution of wealth and employment. One statistic one generally would like to understand is the share of the population which has a wealth below a certain level. The population consists of N individuals. Wealth and labour market status of an individual i is described by the density $p_i(a,z,\tau)$ given an initial condition $(a_i(t), z_i(t))$ drawn from an initial distribution identical for all individuals. The density of each single individual is described by the PDEs in (31). The density of individual wealth (without taking the labour market status into account) is $p_i(a,\tau)$ from (30).

Now define the share of individuals in the entire population with wealth below a certain level a at some point in time $\tau > t$ as $H(a,\tau) \equiv \Sigma_{i=1}^N I(a_i(\tau))/N$ where $I(a_i(\tau))$ is the indicator function taking a value of 1 if $a_i(\tau) < a$ and 0 otherwise. As the $a_i(\tau)$ are identically and independently distributed, the strong

[21] We can not apply prop. 5 as the support of wealth would not be bounded for CARA – at least not as long as linear optimal consumption paths are employed, as is standard in the literature.

law of large numbers holds and we obtain $\lim_{N\rightarrow\infty} H(a,\tau) = \int_{-b/r}^{a} p(x,\tau)\,dx$. In words, the share of individuals in our population with wealth below a is given by the probability that an individual has wealth below a. Computing the derivative of the distribution function gives the density of wealth for the population as a whole, $h(a,\tau) \equiv \frac{d}{da}\int_{-b/r}^{a} p(x,\tau)\,dx = p(a,\tau)$.

When we are interested in wealth distributions for each labour market status individually, we can define $H(a,z,\tau) \equiv \Sigma_{i=1}^{N} I(a_i(\tau), z(\tau))/N$ where the indicator function takes the value of one if $a_i(\tau) < a$ and $z(\tau) = z$. The density is then given by $h(a,z,\tau) = p(a,z,\tau)$.

As has been stressed in the discussion after (28), the initial condition $(a(t), z(t))$ can itself be random. This means that a solution of (31) with an initial distribution for a and z capturing some real world distribution of wealth and employment status provides a prediction how this aggregate distribution evolves over time. We describe our initial conditions by two subdensities, one for employed individuals and one for unemployed individuals, similar to the subdensities in (30),

$$h(a,w,t) = h^{ini}(a,w), \quad h(a,b,t) = h^{ini}(a,b).$$

Empirical information needed to find plausible initial functions (or to estimate them) is the distribution of wealth for employed and unemployed workers. If the share of unemployed workers is x%, the density $h^{ini}(a,w)$ must integrate to x/100, given the property of the subdensity $p(a,w,\tau)$ as shown in (29). If one is primarily interested in understanding the prediction for the aggregate distribution of wealth, any reasonable functions with range $[-b/r, a_w^*]$ and satisfying (29) will do.

7 Conclusion

The objective of this paper was to introduce Fokker-Planck equations as a tool to analyse the dynamics of distributions. We presented the usefulness of these equations by analysing the example of a frictional labour market model that allows individuals to save.

Allowing for savings in standard matching and search models is of high relevance given that individuals would like to self-insure in the presence of uninsurable risk. We derive Keynes-Ramsey rules for optimal consumption, analyse them for different levels of interest rates, illustrate optimal consumption behaviour of workers in a phase-diagram and provide an existence proof for optimal consumption paths.

Fokker-Planck equations (FPEs) for search and matching models are quasi-linear partial differential equations for constant relative risk aversion and linear partial differential equations for constant absolute risk aversion.[22] FPEs describe the density of state variables at each point in time. We therefore do not restrict our analysis to stationary states but can analyse the entire transition path of densities. When transformed into partial differential equations which describe distribution functions,

[22] While we have looked at one example for a search and matching model only, this linearity would survive as long as uncertainty stems only from Poisson processes.

a very intuitive economic interpretation can be provided. The evolution of wealth of workers is subject to stochastic changes due to the transition between the state of employment and unemployment. The evolution of wealth is also determined by deterministic factors (the advection term) which stem from the accumulation of wealth while employed and dissaving while unemployed. The derivation of the FPEs in the appendix is such that the principles behind the various steps are explained in a very accessible way. This should allow to use these tools in other setups as well.

The analysis of distributions of labour market status and wealth in an economy with many agents has also been undertaken. Using a standard law of large numbers, aggregate shares in the population can be linked to individual probabilities. This allows to close the model and obtain general equilibrium results. An additional advantage of FPEs is their promise for fast computation of densities. This should make this approach very suitable for structural estimation.

A problem often encountered in structural estimation with micro data is the lack of model guidance on how to control for aggregate time-series effects. Future work can address this issues by first allowing for explicit transitional dynamics. This would require time varying factor rewards and thereby a generalization of the Keynes-Ramsey rules and of the derivation of the FPEs. Eventually, one should allow for aggregate stochastic disturbances. This would yield exciting and highly promising new results opening up new avenues for estimation.

A Appendix

This appendix contains all proofs and derivations omitted in the main part.

For simple reference in what follows and to simplify notation, define

$$x(a) \equiv c(a, w), \quad y(a) \equiv c(a, b), \tag{A.1}$$

and express the reduced form (21) as

$$\dot{x}(a) = \frac{r - \rho + s\left[\left(\frac{x(a)}{y(a)}\right)^{\sigma} - 1\right]}{ra + w - x(a)} \frac{x(a)}{\sigma}, \tag{A.2a}$$

$$\dot{y}(a) = \frac{r - \rho - \mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra + b - y(a)} \frac{y(a)}{\sigma}. \tag{A.2b}$$

A.1 Proof of prop. 3 concerning Keynes-Ramsey rule

A.1.1 Proof of part (i)

- A local result

We first show that consumption $c(a_w, w)$ rises in time for wealth smaller than but close to a_w^*.

Given that, by ass. 1, the number of sign changes of $\chi'(a)$ in any interval for a of finite length is finite, for any a_0 we can find an $\varepsilon > 0$ such that $\chi(a) \equiv x(a)/y(a)$ is monotonic in $[a_0 - \varepsilon, a_0]$. Exploiting this for a_w^*, whatever the properties of relative consumption, we can always find an ε such that one of the following three cases must holds for $\Omega_\varepsilon \equiv [a_w^* - \varepsilon, a_w^*[$

$$\left.\begin{array}{l}\text{(i)}\\ \text{(ii)}\\ \text{(iii)}\end{array}\right\} \chi'(a)|_{a\in\Omega_\varepsilon} \left\{\begin{array}{c}<\\>\\=\end{array}\right\} 0.$$

Note that we do not make any statement about the derivative in a_w^*. In fact, in case (i) $\chi'(a)|_{a\in a_w^*}$ can be negative or zero, in case (ii), it can be positive or zero.

Lemma 1 *(a) Consumption of employed workers rises over time for a wealth level $a \in \Omega_\varepsilon$ if and only if case (i) holds,*

$$\frac{dc(a_w(\tau), w)}{d\tau} > 0 \text{ for } a_w(\tau) \in \Omega_\varepsilon \Leftrightarrow \text{case (i) holds.}$$

(b) Consumption $c(a_w(\tau), w)$ falls over time for $a_w(\tau) \in \Omega_\varepsilon$ if and only if (ii) holds.

Proof. (a) By (15), $\frac{dc(a_w(\tau),w)}{d\tau} > 0 \Leftrightarrow c(a_w(\tau), w)/c(a_w(\tau), b) > 1/\psi$. As $c(a_w^*, w)/c(a_w^*, b) = 1/\psi$ at a_w^*, as w and b are parameters and using ass. 1, this is a condition on the derivative of relative consumption with respect to wealth a in Ω_ε: $dc(a_w(\tau), w)/d\tau$ is positive for $a_w(\tau) \in \Omega_\varepsilon$ if and only if case (i) holds.

(b) By (15), consumption falls over time if relative consumption lies below $1/\psi$. This can be the case in Ω_ε only if case (ii) holds. ■

Lemma 2 *Relative consumption falls in wealth for $a \in \Omega_\varepsilon$, $\chi'(a)|_{a\in\Omega_\varepsilon} < 0$, i.e. case (i) holds.*

Proof. a) Assume that case (ii) holds, i.e. $\chi'(a)|_{a\in\Omega_\varepsilon} > 0$. Then, by lem. 1, $\frac{dc(a_w(\tau),w)}{d\tau} < 0$ for $a_w(\tau) < a_w^*$. Consumption of unemployed workers would still decrease in time for all wealth levels. In our set Q_v from (23), $\frac{da_w(\tau)}{d\tau} > 0$ and therefore $\frac{dx(a)}{da} < 0$. As $\frac{dc(a_b(\tau),b)}{d\tau} < 0$ and $\frac{da_b(\tau)}{d\tau} < 0$ in Q_v, we know that $\frac{dy(a)}{da} > 0$. As a consequence, $\chi'(a) < 0$. This contradicts the assumption that case (ii) holds and case (ii) can be excluded.

b) Now assume that case (iii) holds, i.e. relative consumption is flat, $\chi'(a)|_{a\in\Omega_\varepsilon\cup a_w^*} = 0$. As $c(a_w^*, w)/c(a_w^*, b) = 1/\psi$, $dc(a_w(\tau), w)/d\tau = 0$ for $a_w(\tau) \in \Omega_\varepsilon$. As $dc(a_b(\tau), b)/d\tau < 0$, relative consumption is not constant – which contradicts the assumption that relative consumption is flat in wealth. As case (iii) is thereby excluded as well, the proof is complete. ■

- A global result

We now complete the proof by a global result on consumption growth.

Lemma 3 *Consumption $c(a_w, w)$ (a) rises in time for all $a < a_w^*$ and (b) decreases in time for all $a > a_w^*$.*

Proof. (a) Imagine to the contrary of "$c(a_w, w)$ rises in time for all $a < a_w^*$" that there is an interval $]\Gamma_1, \Gamma_2[$ with $\Gamma_2 < a_w^*$ such that this is is the last interval before a_w^* where $c(a_w, w)$ falls in time,

$$dc(a_w(\tau), w)/d\tau < 0, \quad \forall\, \Gamma_1 < a_w(\tau) < \Gamma_2 < a_w^*. \tag{A.3}$$

We now proceed as in the proof of lem. 2. As $\frac{da_w(\tau)}{d\tau} > 0$ in Q_v, this would imply that $\frac{dx(a)}{da} < 0$ for $\Gamma_1 < a < \Gamma_2$. We know that $\frac{dy(a)}{da} > 0$ in Q_v. Hence, we would conclude that

$$\chi'(a) < 0, \quad \forall\, \Gamma_1 < a < \Gamma_2. \tag{A.4}$$

By (15), the assumption in (A.3) would hold if and only if relative consumption $\frac{c(a_w,w)}{c(a_w,b)}$ is below $1/\psi$ for $\Gamma_1 < a < \Gamma_2$: $\frac{dc(a_w(\tau),w)}{d\tau} < 0 \Leftrightarrow \frac{c(a_w(\tau),w)}{c(a_w(\tau),b)} < 1/\psi$. As $\frac{x(a)}{y(a)}$ is continuous in wealth by ass. 1 and as case (i) holds by lem. 2, $\frac{x(a)}{y(a)}$ can be smaller than $1/\psi$ only if there is some range $]\Gamma_3, \Gamma_2[$ in which $\chi'(a) > 0$. (An example of such a path is shown in fig. 3.) This is a contradiction to the conclusion in (A.4). Hence, consumption must rise in time for all $a < a_w^*$.

(b) This proof is in analogy to the proof of (a). ■

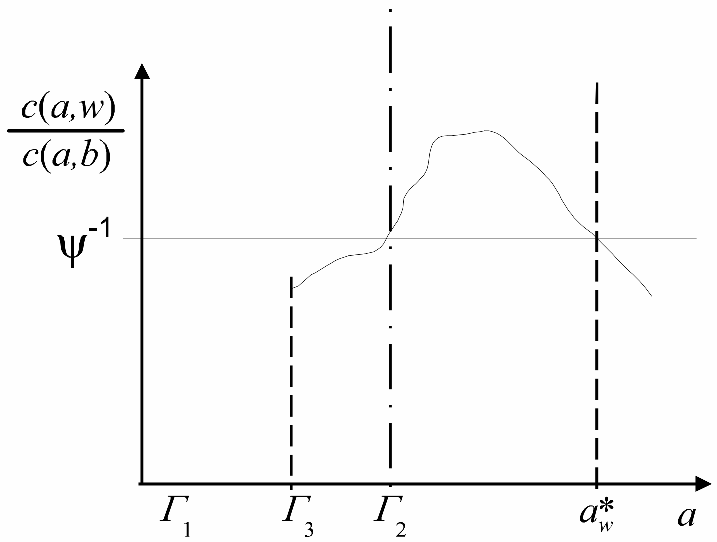

Figure 3 *An example for relative consumption* $\chi(a) \equiv \frac{x(a)}{y(a)}$

A.1.2 Intermediary steps

Before we prove the rest of prop. 3, we need some further intermediary results – which, however, are of some interest in their own right. Given that marginal utility from (4) is positive and decreasing, $u'(c) > 0$ and $u''(c) < 0$, we can establish that $x(a) > y(a)$, i.e. consumption in the state of employment is larger than in the state of unemployment, keeping wealth constant. We prove in passing that the value functions $V(a, z)$ are strictly concave in wealth a.

Lemma 4 *Consumption rises in wealth,* $c_a(a,z) > 0$.

Proof. Prop. 3 (i) shows that $dc(a_w(\tau),w)/d\tau > 0$ in Q_v. As $da_w(\tau)/d\tau > 0$ as well, the derivative $dx(a)/da$ in (21) is positive in Q_v. ■

Lemma 5 *As marginal utility from consumption is positive, the value function* $V(a,z)$ *rises in wealth,* $V_a(a,z) > 0$.

Proof. The first-order condition for optimal consumption is given by (B.3) in the Referees' appendix and reads

$$u'(c(a,z)) = V_a(a,z). \tag{A.5}$$

As marginal utility is positive by (4), the value function rises in wealth. ■

Lemma 6 *As* $u''(c) < 0$ *and as consumption rises in* a *by lemma 4, the value function is strictly concave in* a.

Proof. The partial derivative of the first-order condition with respect to wealth implies

$$u''(c(a,z))\, c_a(a,z) = V_{aa}(a,z). \tag{A.6}$$

As $u''(c(a,z)) < 0$ from the concavity of (4) and $c_a(a,z)$ is positive by lem. 4, $V_{aa}(a,z)$ must be negative. With lem. 5, the value function is strictly concave. ■

Lemma 7 *The shadow price for wealth is higher in the state of unemployment,* $V_a(a,b) > V_a(a,w)$.

Proof. The derivation of the Keynes-Ramsey rule gives us (see app. B.1)

$$\begin{aligned}&(\rho - r)\, V_a(a,z) - s(z)\,[V_a(a,b) - V_a(a,w)] - \mu(z)\,[V_a(a,w) - V_a(a,b)]\\ &= [ra + z - c(a,z)]\, V_{aa}(a,z).\end{aligned}$$

In state $z = w$, this means

$$(\rho - r)\, V_a(a,w) - s(z)\,[V_a(a,b) - V_a(a,w)] = [ra + w - x(a)]\, V_{aa}(a,w). \tag{A.7}$$

Given the region we are interested in (where $ra + w - x(a) > 0$) and given lemma 6, the right-hand side is negative. Hence, the left-hand side must be negative as well. As $(\rho - r)\, V_a(a,w)$ is positive due to $r < \rho$, the second term must be negative. This is the case only for $V_a(a,b) > V_a(a,w)$. ■

Lemma 8 *Consumption of the employed worker is higher than consumption of the unemployed worker,* $x(a) > y(a)$.

Proof. As $V_a(a,b) > V_a(a,w)$, the first-order condition implies $u'(y(a)) > u'(x(a))$. As the marginal utility is decreasing, $x(a) > y(a)$. ■

A.1.3 Proof of parts (ii) and (iii)

(ii) By (17), $dc\left(a_b(\tau), b\right)/d\tau < 0 \Leftrightarrow u'\left(c\left(a_b(\tau), w\right)\right) < \varkappa u'\left(c\left(a_b(\tau), b\right)\right)$ where $\varkappa \equiv 1 - \frac{r-\rho}{\mu} \geq 1$ as $r \leq \rho$. As $u'\left(c\left(a_b(\tau), w\right)\right) < u'\left(c\left(a_b(\tau), b\right)\right)$ with $c\left(a_b(\tau), w\right) > c\left(a_b(\tau), b\right)$ from lem. 8, this condition always holds.

(iii) This follows from solving (19) for relative consumption.

A.2 Proof of theo. 1 - existence of an optimal consumption path

A.2.1 Preliminaries

The natural borrowing limit implies that any solution to (A.2) must satisfy

$$y\left(-b/r\right) = 0. \tag{A.8}$$

In what follows, we will use classical theorems for initial value problems for ODEs. Currently, we have formulated our system (A.2) as a terminal value problem, since the definition of the optimal consumption path in def. 2 uses a terminal condition at the end of the interval $[-b/r, a_w^*]$ under consideration. Using the notation from (A.1), this terminal condition can be written in compact form as

$$\Phi \equiv \Phi_v\left(\hat{a}\right) = \left(\hat{a}, x_v\left(\hat{a}\right), y_v\left(\hat{a}\right)\right). \tag{A.9}$$

Note that Φ depends on v

For ease of notation and to help intuition, we shall now recast the problem into a classical initial value problem, i.e. we will require the value Φ to be attained at the fixed beginning $\tau = 0$ of an interval $[0, \tau^*]$, on which we study the problem. To this end, it is more useful to work with an autonomous system. Hence, we rewrite (A.2) by including $m\left(a\right) = a$ as third variable which "replaces" wealth a, which now purely serves as a parameter, i.e. as the independent variable. By using (A.1), this gives the system

$$\begin{aligned}
\dot{m}(a) &= 1, \\
\dot{x}\left(a\right) &= \frac{r - \rho + s\left[\left(\frac{x(a)}{y(a)}\right)^{\sigma} - 1\right]}{rm\left(a\right) + w - x\left(a\right)} \frac{x\left(a\right)}{\sigma}, \\
\dot{y}\left(a\right) &= \frac{r - \rho - \mu\left[1 - \left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{rm\left(a\right) + b - y\left(a\right)} \frac{y\left(a\right)}{\sigma}.
\end{aligned}$$

Now define $\tau \equiv \hat{a} - a$, $x_1\left(\tau\right) \equiv m\left(\hat{a} - \tau\right)$, $x_2\left(\tau\right) \equiv x\left(\hat{a} - \tau\right)$, $x_3\left(\tau\right) \equiv y\left(\hat{a} - \tau\right).$ Then, $\frac{d}{d\tau} x_1\left(\tau\right) \equiv \dot{x}_1\left(\tau\right) = \frac{d}{d\tau} m\left(\hat{a} - \tau\right) = \frac{d}{d[\hat{a}-a]} m\left(a\right) = -\frac{d}{da} m\left(a\right) = -\dot{m}\left(a\right).$ Doing the same for x and y, the "inverted" autonomous system therefore reads

$$\dot{x}_1(\tau) = -1, \tag{A.10a}$$

$$\dot{x}_2(\tau) = -\frac{r-\rho+s\left[\left(\frac{x_2(\tau)}{x_3(\tau)}\right)^{\sigma}-1\right]}{rx_1(\tau)+w-x_2(\tau)}\frac{x_2(\tau)}{\sigma}, \tag{A.10b}$$

$$\dot{x}_3(\tau) = \frac{r-\rho-\mu\left[1-\left(\frac{x_2(\tau)}{x_3(\tau)}\right)^{\sigma}\right]}{rx_2(\tau)+b-x_3(\tau)}\frac{x_3(a)}{\sigma}, \tag{A.10c}$$

where now $\dot{x}_i$ denotes the derivative of $x_i(\tau)$ with respect to τ, $i = 1, 2, 3$.

Definition 3 *Given (A.10) and for $\tau \geq 0$, let $X(\tau;\Phi) = (x_1(\tau), x_2(\tau), x_3(\tau))$ denote the solution of (A.10) started at $X(0;\Phi) = \Phi \in R_{v,\Psi}$ from (A.9) where $-b/r \leq \hat{a} \leq \frac{\Psi+v-w}{r}$. For later use, we also introduce the notation $x_i(\tau) = x_i(\tau;\Phi)$, $i = 1, 2, 3$.*

By passing from (A.2) to (A.10) we have reverted the time-direction – more precisely, in our setting, the wealth-direction – and turned a non-autonomous system into an autonomous one by including the independent variable as an additional component of the solution. Thus, the curve $a \mapsto (a, x(a), y(a))$ with terminal value $x(\hat{a}) = x_v(\hat{a})$, $y(\hat{a}) = y_v(\hat{a})$ is equal to the curve $\tau \mapsto X(\tau;\Phi)$ with $\Phi = \Phi(\hat{a})$, which is the solution of an initial value problem in the classical sense. However, the parametrization is reverted in the sense that in the former case we start at the left endpoint ("left" in the sense of the smallest value of the a-component) and end in the right endpoint, whereas in the latter case we start at the right endpoint and end in the left one. In particular, the absolute value of the speed along the curve is equal, but the direction is reversed.

A.2.2 Continuity of the solution in initial values

In order to be able to apply classical theorems, we need finite derivatives on the right-hand side of an ODE system. The right-hand side of the ODE (A.2), however, exhibits singularities at the boundary $y = ra+b$ of Q_v. This is of particular importance as the definition of the optimal consumption path in Definition 2 uses $y(-b/r) = 0$ – which lies on this boundary. We obtain finite derivatives by (i) a coordinate transformation and by (ii) (temporarily) reducing the set on which we are interested in a solution by demanding that $y \geq \varepsilon$. We will later show how this reduction can then be removed again by passing $\varepsilon \to 0$.

Lemma 9 *(Coordinate transformation) Let $x(a)$ and $y(a)$ be solutions of (A.2). The mapping $a \mapsto y(a)$ is bijective. Change variables $a = a(y)$ and consider x and a as functions of y. Then*

$$x'(y) \equiv \frac{dx(y)}{dy} = \frac{r-\rho+s\left[\left(\frac{x(y)}{y}\right)^{\sigma}-1\right]}{r-\rho-\mu\left[1-\left(\frac{y}{x(y)}\right)^{\sigma}\right]}\frac{x(y)}{y}\frac{ra(y)+b-y}{ra(y)+w-x(y)}, \tag{A.11a}$$

$$a'(y) \equiv \frac{da(y)}{dy} = \frac{ra(y)+b-y}{r-\rho-\mu\left[1-\left(\frac{y}{x(y)}\right)^{\sigma}\right]}\frac{\sigma}{y}. \tag{A.11b}$$

Proof. Since $\dot{y}(a) > 0$, y is a bijective function of a. As $a'(y) = \frac{1}{\dot{y}(a)}$, we obtain the second equation by inserting (A.2b). The first equation follows from "dividing (A.2a) by (A.2b)". ■

We are going to avoid the singularity at $y(-b/r) = 0$ by temporarily requiring these properties only to hold "up to an arbitrarily small number ε". We do this by considering the domain $R_{\varepsilon,v,\Psi}$ as given in the following

Definition 4 *Fix a numbers $\varepsilon > 0$ and define*

$$R_{\varepsilon,v,\Psi} = R_{v,\Psi} \cap \left\{(a, x, y) \in \mathbb{R}^3 \,|\, y \geq \varepsilon\right\}. \tag{A.12}$$

This definition implies that we temporarily replace the requirement that $y(-b/r) = 0$ by $y(a) = \varepsilon$ for some $-b/r \leq a \leq -b/r + \varepsilon/r$.

Lemma 10 *The right-hand side given in (A.11) is uniformly Lipschitz on $R_{\varepsilon,v,\Psi}$.*

Proof. Consider the right-hand side of (A.11a). The only possible points, where the Lipschitz constant can explode, are when the denominators in the right-hand side become 0 or when a term under a fractional power (i.e. with exponent σ) becomes 0. In $R = R_{\varepsilon,v,\Psi}$, y is uniformly bounded away from 0 and x is uniformly bounded away from $ra+w$. Moreover, note that $r-\rho-\mu\left[1-\left(\frac{y}{x}\right)^\sigma\right] = 0$ if and only if $\left(\frac{y}{x}\right)^\sigma = 1-\frac{r-\rho}{\mu}$. Now $1-\frac{r-\rho}{\mu} > 1$ by the assumption that $r < \rho$. On the other hand, $y < x$, implying that $\left(\frac{y}{x}\right)^\sigma < 1$. Consequently, all the denominators are uniformly bounded away from 0.

For the fractional powers, note that $x/y > 1$ is trivially uniformly bounded away from 0. As $x \leq \Psi$,

$$\frac{y}{x} > \frac{\epsilon}{\Psi}$$

is uniformly bounded away from 0 on $R_{\varepsilon,v,\Psi}$. This shows that (A.11a) is uniformly Lipschitz.

The same arguments show that the right-hand side of (A.11b) is uniformly Lipschitz, too. ■

Since the right hand side of (A.11) is uniformly Lipschitz, we can now apply the classical theory of ODEs. For instance, we have existence and uniqueness of the solution by the Picard-Lindelöf theorem, see Mattheij and Molenaar (2002, th. II.2.3, th. II.3.1). Moreover, the solution will be continuous as a function of the initial value, see, again, Mattheij and Molenaar (2002, th. II.4.7). In the lemma below, we will see how this even implies the corresponding properties for the non-transformed system (A.10).

Lemma 11 *(Continuity in initial values) Consider the set $R = R_{\varepsilon,v,\Psi}$ from (A.12) and the solution $X(\tau;\Phi)$ from Definition 3 with initial condition Φ given in (A.9). The solution $X(\tau;\Phi)$ depends continuously on its initial values Φ. More precisely, there is a constant $L > 0$ and an increasing map $\kappa : [0,\infty[\to [0,\infty[$ (a* modulus of continuity*) with $\lim_{t\searrow 0}\kappa(t) = \kappa(0) = 0$ such that*

$$\|X(\tau_1;\Phi_1) - X(\tau_2;\Phi_2)\| \leq L\|\Phi_1 - \Phi_2\| + \kappa(|\tau_1 - \tau_2|),$$

provided that $\Phi_1, \Phi_2 \in R$ *and* $X(\tau; \Phi_i) \in R$ *for all* $0 \leq \tau \leq \max(\tau_1, \tau_2)$, $i = 1, 2$. *Here,* $\|\cdot\|$ *denotes the Euclidean norm on* $\mathbb{R}^3$.

Proof. By classical results from the theory of ordinary differential equations, see for instance Mattheij and Molenaar (2002, th. II.4.7), the solution of an ODE-system depends continuously on the initial data as long as the right-hand side is uniformly Lipschitz. More precisely, let $Y(\tau; \Phi)$ denote the solution of an ODE with uniformly Lipschitz right-hand side (with Lipschitz constant C), started at $Y(\tau_0; \Phi) = \Phi$, then

$$\|Y(\tau; \Phi_1) - Y(\tau; \Phi_2)\| \leq \exp\left(C(\tau - \tau_0)\right) \|\Phi_1 - \Phi_2\|.$$

Now consider the transformed system $(a(y), x(y))$ from (A.11). By Lemma 10, the right-hand side is uniformly Lipschitz. The solution of (A.11) therefore depends continuously on its initial data (a_0, x_0). It is then obvious that the trajectory $(a(y), x(y), y)$ depends continuously on (a_0, x_0, y_0). As system (A.11) is a reparameterized version of (A.2), the solution $(a, x(a), y(a))$ to (A.2) from def. 2 is also continuous in its boundary conditions – even though the right hand side of (A.2) is not uniformly Lipschitz. Similarly, as (A.10) is just a reparameterization of (A.2), the solution $X(\tau; \Phi)$ to (A.10) from def. 3 is also continuous in its initial condition Φ.

In order to get the estimate, we now consider the ODE (A.10) and note that we only consider it on the compact set $R_{\varepsilon,v,\Psi}$. In the parametrization by y given in (A.11), y is the independent variable, i.e. plays the role of τ in the above estimate. By compactness of $R_{\varepsilon,v,\Psi}$, y only runs through a bounded set, therefore we can rewrite the constant in the above inequality as $\exp(C(y - y_0)) \leq L$ for some suitable $L > 0$.

Given $\Phi \in R_{\varepsilon,v,\Psi}$. Then $a^*_w \leq \frac{\Psi - w + v}{r}$, which implies that the solution $X(\tau; w)$ can only stay inside $R_{\varepsilon,v,\Psi}$ until time $\tau = \frac{\Psi - w + v + b}{r}$, at most. Consider

$$D = \{(\tau, \Phi) \in [0, \infty[\times R_{\varepsilon,v,\Psi} \mid X(\tau; \Phi) \in R_{\varepsilon,v,\Psi}\}.$$

Then D is a closed subset of $\left[0, \frac{\Psi - w + v + b}{r}\right] \times R_{\varepsilon,v,\Psi}$, implying that D is compact. Consequently, $X : D \to R_{\varepsilon,v,\Psi}$ is uniformly continuous, which implies the existence of a modulus of continuity κ with

$$\|X(\tau_1; \Phi_1) - X(\tau_2; \Phi_2)\| \leq \kappa(|\tau_1 - \tau_2| + \|\Phi_1 - \Phi_2\|).$$

The inequality in the lemma then follows by the triangle inequality. ■

A.2.3 Continuity of the first hitting-wealth in initial values

While we have shown in the previous section that the solutions to all systems (A.2), (A.10) and (A.11) are continuous in initial values, this does not automatically imply that the solutions will be continuous on the boundary of the domain we are interested in, in the sense that the place where the solution leaves the domain R might not depend continuously on the initial data. This will now be proved in this section.

In the proofs and also in a later step, we will use the following

Definition 5 *(First hitting-wealth) Consider the set $R_{\varepsilon,v,\Psi}$ from (A.12) and the solution $X(\tau;\Phi)$ to the system (A.10). Consider the path $y(a)$ that corresponds to $x_2(\tau)$ of this solution. Then we define $\hat{a}_{1st} = f(\hat{a})$ as the "first hitting-wealth" (in analogy to first hitting-time), i.e. the wealth level where the path $y(a)$ hits any boundary of $R_{\varepsilon,v,\Psi}$ for the first time. Similarly denote $\tau(\Phi) \equiv \inf\{\tau \geq 0 \,|\, X(\tau;\Phi) \in \partial R_{\varepsilon,v,\Psi}\}$ and $F(\Phi) \equiv X(\tau(\Phi);\Phi)$.*

We know that $\hat{a}_{1st}$ exists because in the set $R_{\varepsilon,v,\Psi}$ the derivatives in (A.10) are well-defined and a solution therefore exists. Notice that $\hat{a}_{1st}$ equals the first component of $F(\Phi(\hat{a}))$.

We also need

Definition 6 *Let $N \subset R_{\varepsilon,v,\Psi}$ with*

$$N = \left\{ \Phi(\hat{a}) \,\middle|\, \hat{a} \in \left[-\frac{b}{r}, \frac{\psi[w-v]-b}{r[1-\psi]} \right] \right\}$$

be the set of all potential initial conditions from (A.9) for a solution in the sense of def. 2. Here we implicitly assume that Ψ is large enough that indeed $N \subset R_{\varepsilon,v,\Psi}$.[23] *Define M as*

$$M = M_1 \cup M_2 \cup M_3 \subset R_{\varepsilon,v,\Psi} \tag{A.13}$$

with

$$\begin{aligned} M_1 &= \{(a,x,y) \in R_{\varepsilon,v,\Psi} \,|\, y = ra + b\}, \\ M_2 &= \{(a,x,y) \in R_{\varepsilon,v,\Psi} \,|\, a = -b/r\}, \\ M_3 &= \{(a,x,y) \in R_{\varepsilon,v,\Psi} \,|\, y = \varepsilon\}. \end{aligned}$$

This set will turn out to be the set of all potential first hitting-wealths.

Since we know that $x > y$, the trajectory will not hit the boundary of R at the part $\{x = y\}$. Therefore, we have the

Corollary 1 *$F : N \to M$ is a well-defined map, i.e. for every $\Phi \in N$, the corresponding solution path $X(\tau;\Phi)$ exists and stays in $R_{\varepsilon,v,\Psi}$ until it finally hits M (and no other boundary of $R_{\varepsilon,v,\Psi}$).*

Before formulating the main lemma of this section, let us first derive a simple bound on the derivative $\dot{y}(a)$ of the consumption of the unemployed.

Lemma 12 *For (a,x,y) in the interior of Q_v from (23), we have*

$$\dot{y}(a) \geq \frac{r-\rho}{ra+b-y(a)} \frac{y(a)}{\sigma}.$$

[23] This is the only necessary condition on Ψ for the construction to work. In the sequel, we shall assume this condition without further notice.

Proof. By (A.2b) we have

$$\dot{y}\left(a\right)=\frac{r-\rho-\mu\left[1-\left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra+b-y\left(a\right)}\frac{y\left(a\right)}{\sigma}$$
$$=\left(\frac{r-\rho}{ra+b-y\left(a\right)}-\frac{\mu\left[1-\left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra+b-y\left(a\right)}\right)\frac{y\left(a\right)}{\sigma}>\frac{r-\rho}{ra+b-y\left(a\right)}\frac{y\left(a\right)}{\sigma}.$$

The last inequality follows from the fact that $\frac{\mu\left[1-\left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra+b-y(a)}$ is negative (and therefore $-\frac{\mu\left[1-\left(\frac{y(a)}{x(a)}\right)^{\sigma}\right]}{ra+b-y(a)}$ is positive) as $ra+b-y\left(a\right)$ is negative in the interior of Q_v. ■

The key result in this section is presented in

Lemma 13 *The map $F:N\rightarrow M$ is continuous.*

Proof. We need to prove that for every $\Phi\in N$ and every $\delta>0$ there is an $\eta>0$ such that

$$\|\Phi_0-\Phi\|<\eta\Longrightarrow\|F(\Phi_0)-F(\Phi)\|<\delta. \tag{A.14}$$

We start the proof by fixing Φ_0, $\Phi\in N$ such that $\|\Phi_0-\Phi\|<\eta$ for some $\eta>0$. Let us first *assume* that $\tau(\Phi_0)\leq\tau(\Phi)$. By the triangle inequality and Lemma 11, we have

$$\begin{aligned}\|X(\tau(\Phi_0);\Phi_0)-X(\tau(\Phi);\Phi)\| &\leq \|X(\tau(\Phi_0);\Phi_0)-X(\tau(\Phi_0);\Phi)\|+\\ &\quad+\|X(\tau(\Phi_0);\Phi)-X(\tau(\Phi);\Phi)\|\\ &\leq L_1\|\Phi_0-\Phi\|+\kappa(|\tau(\Phi_0)-\tau(\Phi)|),\end{aligned} \tag{A.15}$$

for a constant $L_1>0$ and the modulus of continuity κ. In order to get an estimate for $|\tau(\Phi_0)-\tau(\Phi)|$, we have to distinguish between three different cases.

Case (i): $F(\Phi_0)\in M_1$.
By Lemma 12, there are constants $L_2,\ell_2>0$ such that $\dot{y}\geq L_2$ for $|y-(ra+b)|\leq\ell_2$. More precisely, we can choose $\ell_2>0$ freely and obtain the bound for $L_2=\frac{1}{\ell_2}\frac{(\rho-r)\varepsilon}{\sigma}$. If $L_1\eta\leq\ell_2$, we can bound the absolute value of the derivative of $x_3(\tau;\Phi)$ from below by L_2 (for $t\geq\tau(\Phi_0)$). This implies that the path $X(\tau;\Phi)$ hits M_1 before time $\tau(\Phi_0)+\tau$ for

$$\tau(L_2-r)=\ell_2\iff\tau=\frac{\ell_2}{L_2-r},$$

unless it hits another boundary of $R_{\varepsilon,v,\Psi}$ before that. Inserting into (A.15), this gives the estimate

$$\|F(\Phi_0)-F(\Phi)\|\leq L_1\eta+\kappa\left(\frac{\ell_2}{L_2-r}\right).$$

Choosing $\ell_2=L_1\eta$, the bound is smaller than δ provided that

$$\kappa\left(\frac{L_1}{\frac{C}{L_1\eta}-r}\eta\right)+L_1\eta<\delta, \tag{A.16}$$

where $C \equiv \frac{(\rho-r)\varepsilon}{\sigma}$. Note that the left hand side in (A.16) converges to zero for $\eta \to 0$, therefore we can find an $\eta_0(\delta) > 0$ (only depending on the constants C, L_1 and r and the modulus of continuity κ, but not on Φ_0 or Φ) such that the desired inequality (A.14) holds for $\eta < \eta_0$. We have tacitly assumed that $L_2 = C/\ell_2 = \frac{C}{L_1\eta} > r$, which can be realized by choosing η small enough.

Case (ii): $F(\Phi_0) \in M_2$.
Let $\hat{a}$ denote the first component of Φ, and $\hat{a}_0$ the first component of Φ_0. Note that $x_1(\tau;\Phi) = \hat{a} - \tau$, for every $\tau \geq 0$. Since $X(\tau(\Phi_0);\Phi_0) \in M_2$, we have $-b/r = x_1(\tau(\Phi_0);\Phi_0) = \hat{a}_0 - \tau(\Phi_0)$, implying that $\tau(\Phi_0) = \hat{a}_0 + b/r$. On the other hand, $x_1(\tau(\Phi);\Phi) \geq -b/r$, implying that $\tau(\Phi) \leq \hat{a} + b/r$. Combining these two results, we obtain

$$|\tau(\Phi_0) - \tau(\Phi)| = \tau(\Phi) - \tau(\Phi_0) \leq \hat{a} - \hat{a}_0 \leq \|\Phi_0 - \Phi\|.$$

Consequently, the inequality (A.15) implies

$$\|F(\Phi_0) - F(\Phi)\| \leq L_1 \|\Phi_0 - \Phi\| + \kappa(\|\Phi_0 - \Phi\|) \leq L_1\eta + \kappa(\eta),$$

and (A.14) holds for η small enough such that

$$L_1\eta + \kappa(\eta) < \delta. \tag{A.17}$$

Case (iii): $F(\Phi_0) \in M_3$.
Since $x_3(\tau(\Phi_0);\Phi_0) = \varepsilon$, we have $0 \leq x_3(\tau(\Phi_0);\Phi) - \varepsilon \leq L_1\eta$. By Lemma 12, we can find a constant $L_3 > 0$ such that $\dot{y} \geq L_3$ on $R_{\varepsilon,v,\Psi}$ – note that L_3 depends on ε. Thus, $X(s;\Phi)$ will hit the boundary M_3 before time $\tau(\Phi_0) + \tau$ with $\tau = L_1\eta/L_3$, unless it hits another boundary of $R_{\varepsilon,v,\Psi}$ before. In any case, $|\tau(\Phi_0) - \tau(\Phi)| \leq L_1\eta/L_3$, and we obtain

$$\|F(\Phi_0) - F(\Phi)\| \leq L_1\eta + \kappa\left(\frac{L_1}{L_3}\eta\right),$$

and (A.14) is satisfied for

$$L_1\eta + \kappa\left(\frac{L_1}{L_3}\eta\right) < \delta. \tag{A.18}$$

Choosing η small enough that both (A.16) and (A.17) and (A.18) are satisfied, settles the proof for $\tau(\Phi_0) \leq \tau(\Phi)$. Notice that none of the conditions (A.16), (A.17) and (A.18) depends on Φ_0. Therefore, in the other case $\tau(\Phi_0) \geq \tau(\Phi)$, we can just revert the rôles of Φ and Φ_0 and obtain the same results in cases (i), (ii) and (iii). ■

A.2.4 Existence of a solution

This section proves our main result formulated in Theorem 1.

Proof. Fix some $\varepsilon > 0$ and consider $R_{\varepsilon,v,\Psi}$. By an intermediate value theorem applied to $F : N \to M$, we will obtain a point or points $\Phi \in N$ such that $F(\Phi) \in M_3$ as used in (A.13), i.e. $x_3(\tau(\Phi);\Phi) = \varepsilon$ provided that we can show the existence of points (that could be called upper and lower bounds) $\Phi_v^{\min}, \Phi_v^{\max} \in N$ with $F(\Phi_v^{\min}) \in M_2$ and $F(\Phi_v^{\max}) \in M_1$. (Note that $F = F_\varepsilon$ and all the $M_i = M_i(\varepsilon)$, $i = 1, 2, 3$, depend on ε and v, but not on Ψ, provided that Ψ is large enough.)

Choose

$$\Phi_v^{\min} = \Phi(-b/r) = (-b/r, w-b-v, \psi\,[w-b-v]), \quad \Phi_v^{\max} = \Phi\left(\frac{\psi(w-v)-b}{(1-\psi)r}\right).$$

By construction, both $\Phi_v^{\min}$ and $\Phi_v^{\max}$ are contained in N. Moreover, we trivially have $F_\varepsilon(\Phi_v^{\min}) \in M_2(\varepsilon)$, $F_\varepsilon(\Phi_v^{\max}) \in M_1(\varepsilon)$ for every $\varepsilon > 0$ small enough. Note, in particular, that Lemma 13 also implies continuity of F in the boundary points $\Phi_v^{\min}$ and $\Phi_v^{\max}$ of N. Therefore, the image set $F_\varepsilon(N)$ is a connected set, with non-empty intersection with both M_1 and M_2. Since the distance

$$\operatorname{dist}(M_1, M_2) = \inf\left\{\|\Phi_1 - \Phi_2\| \mid \Phi_1 \in M_1,\, \Phi_2 \in M_2\right\} = \frac{\varepsilon}{r} > 0,$$

we may conclude that $F_\varepsilon(N) \cap M_3(\varepsilon) \neq \emptyset$. This establishes that there must be a Φ such that $F_\varepsilon(\Phi) \in M_3$. In words, there is an initial condition $\Phi(\hat{a})$ such that the path $(a, x(a), y(a))$ hits the boundary at $y = \varepsilon$.

Now define

$$N_3(\varepsilon) \equiv F_\varepsilon^{-1}(M_3(\varepsilon)) = \left\{\Phi \in N \mid F_\varepsilon(\Phi) \in M_3(\varepsilon)\right\}.$$

By continuity of $F_\varepsilon : N \to M(\varepsilon)$, the bounded set $N_3(\varepsilon)$ is closed and thus compact. Moreover, the family $(N_3(\varepsilon))_{\varepsilon>0}$ is directed in the sense that

$$0 < \varepsilon_2 < \varepsilon_1 \Longrightarrow N_3(\varepsilon_2) \subset N_3(\varepsilon_1).$$

By standard results from topology, the intersection of a directed family of non-empty, compact sets is non-empty, i.e.

$$N_3(0) \equiv \bigcap_{\varepsilon>0} N_3(\varepsilon) \neq \emptyset.$$

Indeed, take a decreasing sequence $(\varepsilon_n)_{n\geq 1}$ of positive numbers converging to zero. For every n choose some $\Phi_n \in N_3(\varepsilon_n)$. By compactness of the largest set $N_3(\varepsilon_1)$, we can find a subsequence n_k such that $(\Phi_{n_k})_{k\geq 1}$ converges to some Φ. Note that $\Phi \in N_3(\varepsilon_{n_k})$ for every k, since $\Phi = \lim_{l\to\infty,\, l\geq k} \Phi_{n_l}$ and each such Φ_{n_l} lies in the closed set $N_3(\varepsilon_{n_k})$. Now choose any $\varepsilon > 0$ and pick a k such that $\varepsilon_{n_k} < \varepsilon$. Then $\Phi \in N_3(\varepsilon_{n_k}) \subset N_3(\varepsilon)$, implying that $\Phi \in \bigcap_{\varepsilon>0} N_3(\varepsilon)$.

We claim that every element $\Phi \in N_3(0)$ corresponds to an aTSS. Indeed, the path $(a, x(a), y(a))$ with terminal value $(\hat{a}, \hat{x}, \hat{y}) = \Phi$ (corresponding to the path $X(\tau; \Phi)$) satisfies the ODE (A.2) on $]-b/r, \hat{a}]$. Moreover, it starts at N by construction, and for every $\varepsilon > 0$, it takes on the value ε somewhere on the interval $]-b/r, -b/r+\varepsilon[$. Thus, using monotonicity of y, we may conclude that

$$\lim_{a \searrow -b/r} y(a) = 0.$$

This establishes that there is an initial condition $\Phi(\hat{a})$ such that the path $y(a)$ hits the boundary at $y = 0$ in the sense that $y(-b/r) = 0$. ■

Note that it is essential for the proof of Theorem 1 that the trajectory $X(\tau;\Phi)$ – or, equivalently, $(a, x(a), y(a))$ – does not depend on ε, which only determines "how long" we observe the trajectory. This means that we observe the trajectory $X(\tau;\Phi)$ for $0 \leq \tau \leq \tau(\Phi)$, with the hitting time $\tau(\Phi)$ obviously depending on ε. Therefore, we can, for fixed $\Phi \in N_3(0)$, easily take the limit $\varepsilon \to 0$, which means that we take the limit in $\tau(\Phi)$, but do not change the trajectory itself. As a consequence, the ODE is automatically satisfied for the limit, at least for $0 \leq \tau < \lim_{\varepsilon\to 0} \tau(\Phi)$.

Let us illustrate why we had to use the specific properties of the dynamic system (A.10) in the proof of lem. 13. Continuity in initial conditions does not imply continuity of "first hitting values" in general. Indeed, the first hitting times are inherently non-continuous functionals, even if both the paths and the set, which determines the hitting times, are smooth.

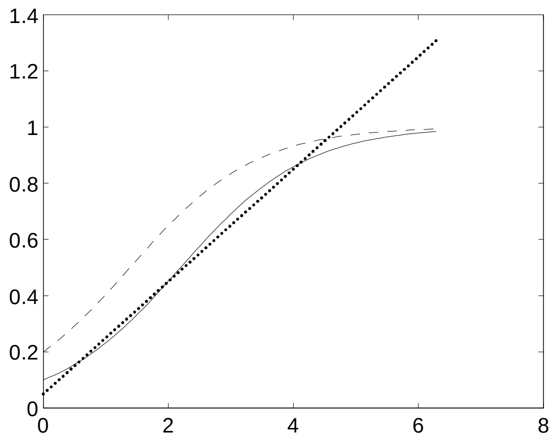

Figure 4 *Non-continuity of the first hitting time*

To see this most clearly, consider the differential equation $\dot{z}(t) = (1 - z(t))\, z(t)$ whose solution is $z(t) = \left(1 + \left(z_0^{-1} - 1\right) e^{-t}\right)^{-1}$. This solution is continuous in the initial level z_0 (for $z_0 > 0$ which we assume) and the solution is plotted for $z_0 \in \{0.1, 0.2\}$ in fig. 4. Now consider the first-hitting time on the straight line $0.05 + t/5$ as drawn. Obviously, this time is not continuous in the initial values z_0.

A.3 Deriving the Fokker-Planck equations (31)

This section picks up after sect. 5.2.1 and derives the Fokker-Planck equations of the wealth-employment process $(a(t), z(t))$. The derivation is in great detail as this facilitates applications for other purposes. Before we go through individual steps, here is the general idea. Step 1: We start with some function f having as arguments the variables whose density we would like to understand. We compute the differential of this function in the usual way and also compute its expected change. Step 2: The starting point here is Dynkin's formula. This formula, intuitively speaking, gives the expected value of some function f, whose arguments are the random variables we are interested in, as the sum of the current value of f plus the integral over expected future changes of f. The expected change of f is expressed by using the density of our random variables. The Dynkin formula is differentiated with respect to time. Step 3: By using integration by parts or the adjoint operator, we get an expression for the change of the expected value of f. Step 4: A different expression for this change of the expected value can be obtained by starting from the expected value and differentiating it. Step 5: Equating the two gives the differential equations for the density.

It should be kept in mind that this approach is relatively general and not at all restricted to our system. As long as there is one to several stochastic processes described by stochastic differential equations, this approach can be used to obtain a description of the densities. It does in particular not matter what the source of uncertainty is. Brownian motion works just as well as Poisson processes, as would a combination of the two or the more general Levy processes.

A.3.1 The expected change of some function f

Assume there is a function f having as arguments the state variables a and z. This function has a bounded support S, i.e. $f(a,z) = 0$ outside this support.[24] Heuristically, the differential of this function, using a change of variable formula,[25] gives

$$\begin{aligned} df\left(a\left(\tau\right),z\left(\tau\right)\right) = {} & f_a\left(.\right)\left\{ra\left(\tau\right)+z\left(\tau\right)-c\left(a\left(\tau\right),z\left(\tau\right)\right)\right\}d\tau \\ & +\left\{f\left(a\left(\tau\right),z\left(\tau\right)+\Delta\right)-f\left(a\left(\tau\right),z\left(\tau\right)\right)\right\}dq_\mu \\ & +\left\{f\left(a\left(\tau\right),z\left(\tau\right)-\Delta\right)-f\left(a\left(\tau\right),z\left(\tau\right)\right)\right\}dq_s . \end{aligned}$$

Due to the state-dependent arrival rates (see tab. 1), only one Poisson process is active at a time.

When we are interested in the expected change, we need to form expectations.[26] Applying the conditional expectations operator E_τ and dividing by $d\tau$ yields the heuristic equation

$$\begin{aligned} \frac{E_\tau df\left(.\right)}{d\tau} = {} & f_a\left(.\right)\left\{ra\left(\tau\right)+z\left(\tau\right)-c\left(a\left(\tau\right),z\left(\tau\right)\right)\right\} \\ & +\mu\left(z\left(\tau\right)\right)\left[f\left(a\left(\tau\right),z\left(\tau\right)+\Delta\right)-f\left(a\left(\tau\right),z\left(\tau\right)\right)\right] \\ & +s\left(z\left(\tau\right)\right)\left[f\left(a\left(\tau\right),z\left(\tau\right)-\Delta\right)-f\left(a\left(\tau\right),z\left(\tau\right)\right)\right] \end{aligned} \tag{A.19}$$

In what follows, we denote this expression by

$$\mathcal{A}f\left(a\left(\tau\right),z\left(\tau\right)\right) \equiv \frac{E_\tau df\left(a\left(\tau\right),z\left(\tau\right)\right)}{d\tau} \tag{A.20}$$

[24] We can make this assumption without any restriction. As we will see below, this function will not play any role in the determination of the actual density.

[25] There are formal derivations of this equation in mathematical textbooks like Protter (1995). For a more elementary presentation, see Wälde (2010, part IV).

[26] We view $a(\tau)$ and $z(\tau)$ with $\tau \geq t$ as two stochastic processes which start in t and where initial conditions $a(t)$ and $z(t)$ can be random variables. We therefore form expectations about df by using the unconditional expectations operator E as the randomness of initial values are then also taken into account. This is useful for its generality and also when it comes to applications (see sect. 5.2.3 on initial conditions and especially distributions).

which is, more precisely, the infinitesimal generator $\mathcal{A}$ defined by

$$\mathcal{A}f(a,z) = \lim_{\epsilon \searrow 0} \frac{E\left(f(z(\tau+\epsilon), a(\tau+\epsilon))|z(\tau)=z,\ a(\tau)=a\right) - f(a,z)}{\epsilon}.$$

Notice that $\mathcal{A}f(a,z)$ does not depend on τ, because the Markov-process $(a(\tau), z(\tau))$ is time-homogeneous. We understand $\mathcal{A}$ as an operator mapping functions (in a and z) to other such functions. Moreover, note that all test-functions, i.e. C^∞ functions of bounded support, are in the domain of the operator $\mathcal{A}$, i.e. the domain of all functions f such that the above limit exists (for all a and z).

A.3.2 Dynkin's formula and its manipulation

To abbreviate notation, we now define $x(\tau) \equiv (a(\tau), z(\tau))$. The expected value of our function $f(x(\tau))$ is by Dynkin's formula (e.g. Yuan and Mao, 2003) given by

$$Ef(x(\tau)) = Ef(x(t)) + \int_t^\tau E(\mathcal{A}f(x(s)))\, ds. \tag{A.21}$$

To understand this equation, use the definition in (A.20) and formally write it as

$$Ef(x(\tau)) = Ef(x(t)) + \int_t^\tau \frac{Edf(x(s))}{ds} ds = Ef(x(t)) + \int_t^\tau Edf(x(s)).$$

Intuitively speaking, Dynkin's formula says that the expected value of $f(x(\tau))$ is the expectation for the current value, $Ef(x(t))$ (given that we allow for a random initial condition $x(t)$), plus the "sum of" expected future changes, $\int_t^\tau Edf(x(s))$.

Let us now differentiate (A.21) with respect to time τ and find

$$\frac{\partial}{\partial \tau} Ef(x(\tau)) = \frac{\partial}{\partial \tau} \int_t^\tau E(\mathcal{A}f(x(s)))\, ds = E(\mathcal{A}f(x(\tau))), \tag{A.22}$$

where the first equality used that $Ef(x(t))$ is a constant and pulled the expectations operator into the integral. This equation says the following: We form expectations in t about $f(x(\tau))$. We now ask how this expectation changes when τ moves further into the future, i.e. we look at $\frac{\partial}{\partial \tau} E[f(x(\tau))]$. We see that this change is given by the expected change of $f(x(\tau))$, where the change is $\mathcal{A}f(x(\tau))$.

We now introduce the densities we defined in sect. 5.2.1. The expectation operator E in (A.22) integrates over all possible states of $x(\tau)$. When we express this joint density as $p(a,z,\tau) \equiv p(a,\tau|z)\, p_z(\tau)$, we can write (A.22) as

$$\begin{aligned} \frac{\partial}{\partial \tau} Ef(x(\tau)) &= E(\mathcal{A}f(x(\tau))) \\ &= p_w(\tau) \int_{-\infty}^{\infty} \mathcal{A}f(a,w)\, p(a,\tau|w)\, da + p_b(\tau) \int_{-\infty}^{\infty} \mathcal{A}f(a,b)\, p(a,\tau|b)\, da. \end{aligned}$$

Now pull $p_w(\tau)$ and $p_b(\tau)$ back into the integral and use $p(a,z,\tau) \equiv p(a,\tau|z)\, p_z(\tau)$ again for $z = w$ and $z = b$. Then

$$\begin{aligned} \frac{\partial}{\partial \tau} Ef(x(\tau)) &= \int_{-\infty}^{\infty} \mathcal{A}f(a,w)\, p(a,w,\tau)\, da + \int_{-\infty}^{\infty} \mathcal{A}f(a,b)\, p(a,b,\tau)\, da \\ &\equiv \phi_w + \phi_b. \end{aligned} \tag{A.23}$$

A.3.3 The adjoint operator and integration by parts

This is now the crucial step in obtaining a differential equation for the density. It consists in applying an integration by parts formula which allows to move the derivatives in $\mathcal{A}f(x(\tau))$ into the density $p(x,\tau)$. Let us briefly review this method, without getting into technical details. Given two functions $f,g:\mathbb{R}\to\mathbb{R}$ and two fixed real numbers $c<d$, the factor rule of differentiation

$$d(f(x)\cdot g(x)) = df(x)\cdot g(x) + f(x)\cdot dg(x) \tag{A.24}$$

implies that $f(d)g(d)-f(c)g(c)=\int_c^d f'(x)g(x)dx+\int_c^d f(x)g'(x)dx$, a formula referred to as partial integration rule. In particular, it also holds for $c=-\infty$ and $d=+\infty$, if the function evaluations are understood as limits for $c\to-\infty$ and $d\to+\infty$, respectively. If f has bounded support, i.e. is equal to zero outside a fixed bounded set, then the function evaluations at $\pm\infty$ vanish and we get

$$\int_{-\infty}^{+\infty} f'(x)g(x)dx = -\int_{-\infty}^{+\infty} f(x)g'(x)dx. \tag{A.25}$$

We now apply (A.25) to equation (A.23). We can do this as the expressions in (A.23) "lost" all stochastic features. To this end, insert the definition of $\mathcal{A}$ given in (A.20) together with (A.19) into (A.23). To avoid getting lost in long expressions, we look at the both integrals in (A.23) in turn. For the second, observe that

$$\mathcal{A}f(a,b) = f_a(.)\{ra+b-c(a,b)\} + \mu\left[f(a,w)-f(a,b)\right],$$

i.e. the term with s in (A.19) is missing given that we are in state b. Hence,

$$\begin{aligned}\phi_b &= \int_{-\infty}^{\infty}\left[f_a(a,b)\{ra+b-c(a,b)\}+\mu\left[f(a,w)-f(a,b)\right]\right]p(a,b,\tau)\,da\\ &= \int_{-\infty}^{\infty} f_a(a,b)\{ra+b-c(a,b)\}p(a,b,\tau)\,da\\ &\quad+\int_{-\infty}^{\infty}\mu\left[f(a,w)-f(a,b)\right]p(a,b,\tau)\,da.\end{aligned}$$

Now integrate by parts. As this integral shows, we only need to integrate by parts for the f_a term. The rest remains untouched. This gives with (A.25), where $g(x)$ stands for $\{ra+b-c(a,b)\}p(a,b,\tau)$ and x for a,

$$\begin{aligned}\phi_b = &-\int_{-\infty}^{\infty} f(a,b)\left[\left\{r-\frac{\partial}{\partial a}c(a,b)\right\}p(a,b,\tau)+\{ra+b-c(a,b)\}\frac{\partial}{\partial a}p(a,b,\tau)\right]da\\ &+\int_{-\infty}^{\infty}\mu\left[f(a,w)-f(a,b)\right]p(a,b,\tau)\,da.\end{aligned} \tag{A.26}$$

Now look at the first integral of (A.23). After similar steps (as the principle is the same, we replace b by w and the arrival rate μ by s in the last equation), this reads

$$\begin{aligned}\phi_w = &-\int_{-\infty}^{\infty} f(a,w)\left[\left\{r-\frac{\partial}{\partial a}c(a,w)\right\}p(a,w,\tau)+\{ra+w-c(a,w)\}\frac{\partial}{\partial a}p(a,w,\tau)\right]da\\ &+\int_{-\infty}^{\infty}s\left[f(a,b)-f(a,w)\right]p(a,w,\tau)\,da.\end{aligned} \tag{A.27}$$

Summarizing, we find

$$\frac{\partial}{\partial \tau} Ef\left(x\left(\tau\right)\right) = \phi_w + \phi_b$$

$$= \int_{-\infty}^{\infty} f\left(a,w\right) \left[-\left\{ r - \frac{\partial}{\partial a} c\left(a,w\right) \right\} p\left(a,w,\tau\right) - \left\{ ra + w - c\left(a,w\right) \right\} \frac{\partial}{\partial a} p\left(a,w,\tau\right) \right] da$$
$$+ \int_{-\infty}^{\infty} s\left[f\left(a,b\right) - f\left(a,w\right) \right] p\left(a,w,\tau\right) da$$
$$+ \int_{-\infty}^{\infty} f\left(a,b\right) \left[-\left\{ r - \frac{\partial}{\partial a} c\left(a,b\right) \right\} p\left(a,b,\tau\right) - \left\{ ra + b - c\left(a,b\right) \right\} \frac{\partial}{\partial a} p\left(a,b,\tau\right) \right] da$$
$$+ \int_{-\infty}^{\infty} \mu\left[f\left(a,w\right) - f\left(a,b\right) \right] p\left(a,b,\tau\right) da. \tag{A.28}$$

A.3.4 The expected value again

Let us now derive the second expression for the change in the expected value. By definition, and as an alternative to the Dynkin formula (A.21), we have

$$Ef\left(x\left(\tau\right)\right) = \int_{-\infty}^{\infty} f\left(a,b\right) p\left(a,b,\tau\right) da + \int_{-\infty}^{\infty} f\left(a,w\right) p\left(a,w,\tau\right) da. \tag{A.29}$$

When we differentiate this expression with respect to time, we get

$$\frac{\partial}{\partial \tau} Ef\left(x\left(\tau\right)\right) = \int_{-\infty}^{\infty} f\left(a,b\right) \frac{\partial}{\partial \tau} p\left(a,b,\tau\right) da$$
$$+ \int_{-\infty}^{\infty} f\left(a,w\right) \frac{\partial}{\partial \tau} p\left(a,w,\tau\right) da. \tag{A.30}$$

Note that we can use

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} f\left(a,z\right) p\left(a,z,\tau\right) da = \int_{-\infty}^{\infty} f\left(a,z\right) \frac{\partial}{\partial \tau} p\left(a,z,\tau\right) da$$

as z and a inside this integral are no longer functions of time.

A.3.5 Equating the two expressions

We now equate (A.28) with (A.30). Collecting terms belonging to $f\left(a,w\right)$ and $f\left(a,b\right)$ gives

$$\int_{-\infty}^{\infty} f\left(a,w\right) \varphi_w da + \int_{-\infty}^{\infty} f\left(a,b\right) \varphi_b da = 0, \tag{A.31}$$

where

$$\varphi_w \equiv -\left\{ r - \frac{\partial}{\partial a} c\left(a,w\right) + s \right\} p\left(a,w,\tau\right) - \left\{ ra + w - c\left(a,w\right) \right\} \frac{\partial}{\partial a} p\left(a,w,\tau\right)$$
$$+ \mu p\left(a,b,\tau\right) - \frac{\partial}{\partial \tau} p\left(a,w,\tau\right)$$

and

$$\varphi_b \equiv -\left\{r - \frac{\partial}{\partial a}c(a,b) + \mu\right\} p(a,b,\tau) - \{ra + b - c(a,b)\} \frac{\partial}{\partial a} p(a,b,\tau)$$
$$+ sp(a,w,\tau) - \frac{\partial}{\partial \tau} p(a,b,\tau).$$

Obviously, the above equation is satisfied if

$$\varphi_b = \varphi_w = 0. \tag{A.32}$$

These are the Fokker-Planck equations used in (31).

It is easy to see that the integral equation can only be satisfied for all functions f if these Fokker-Planck equations are satisfied. Indeed, assume that $\varphi_b > 0$ on an interval $I = [d - \epsilon, d + \epsilon]$. One can find a non-negative function f smooth in a such that $f(a,w) = 0$ for all a and

$$f(a,b) = \begin{cases} 1, & a \in [d - \epsilon/2, d + \epsilon/2], \\ 0, & a \in] - \infty, d - \epsilon] \cup [d + \epsilon, \infty[. \end{cases}$$

Inserting this test function into the integral equation gives

$$\int_{-\infty}^{\infty} f(a,w)\, \varphi_w da + \int_{-\infty}^{\infty} f(a,b)\, \varphi_b da = 0 + \int_{d-\epsilon}^{d+\epsilon} f(a,b)\, \varphi_b da > 0$$

by construction. Therefore, $\varphi_b = 0$ has to hold for all $a \in \mathbb{R}$, and similarly for φ_w.

B Referees' appendix

This appendix is available upon request.

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