Poverty and Labour Market Policies in a Stochastic and Dynamic World

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Abstract

Poverty is a persistent problem of any society. Using a relative poverty line of 50% of the median disposable income, the average relative poverty share for the 27 member states of the Organisation for Economic Co-operation and Development (OECD) was 11.1% in the late 2000s. Here, the U.S. had the fourth highest rate of 17.3% (OECD, 2011). Causes for poverty and its extend are multifaceted and vary in between countries. Nevertheless, recent studies as for example Oxley et al. (2000) stress the impact of tax and transfer systems on poverty.

The objective of this thesis is to analyse the link between public and private provisions to poverty. At first, a new quantitative method, the so-called Fokker-Planck equation, will be introduced, which allows the analysis of distributional aspects of stochastic dynamic models. Subsequently, using the National Longitudinal Survey of Youth 1979, the evolution of wealth-poverty in the U.S. will be analysed. Therefore, a numerical solution of a simple labour market model, motivated by Bayer and Wälde (2013), is used. A calibrated version of this model partially matches the poverty evolution over 22 years. Results show that in such a setup the reduction of unemployment benefits can lead to less poverty in terms of poverty gap and head count.

In jeder Gesellschaft stellt Armut ein dauerhaftes Problem dar. Definiert man die Armutsgrenze als 50% des verfügbaren Einkommens des Medians, lag in den späten 2000ern die durchschnittliche relative Armutsquote aller 27 Mitgliedstaaten der Organisation für wirtschaftliche Zusammenarbeit und Entwicklung (OECD) bei 11,1%. Hierbei lag die Quote in den USA bei 17,3%, dem vierthöchsten Wert unter allen OECD-Staaten (OECD, 2011). Ursachen und Ausmaße von Armut sind sehr vielfältig und unterschiedlich für verschiedene Länder. Grundsätzlich, wie z.B. beschrieben in Oxley et al. (2000), ist der Einfluss von Steuersystemen und Sozialversicherungen auf das Ausmaß von Armut bekannt.

Ziel dieser Arbeit ist die genauere Analyse des Zusammenhangs von öffentlichen und privaten Vorsorgemaßnahmen auf die Armutsverhältnisse. Hierzu wird zunächst eine neue quantitative Methode, die sogenannte Fokker-Planck Gleichung, eingeführt. Diese Gleichung erlaubt die Analyse von Verteilungen in allgemeinen stochastischen und dynamischen Modellen. Anschließend wird die Entwicklung der amerikanischen Vermögensverteilung, beschrieben im "National Longitudinal Survey of Youth 1979", analysiert. Hierbei wird ein einfaches Arbeitsmarktmodell, basierend auf Bayer und Wälde (2013), numerisch gelöst und bezüglich der Armutsentwicklung in den USA kalibriert. Analysen zeigen, dass in diesem Modellrahmen eine Minderung der Arbeitslosenhilfen zu einer Reduktion der Armut (gemessen anhand der Armutsquote und der Armutslücke) führt.

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Contents

Chapter 5

Description of Distributional Properties using the Method of Characteristics

by Tobias Nagel[†]

5.1 Introduction

Motivation - Dynamic frameworks are used in economic growth models, labour market models and many more. The dynamics are commonly modelled with the help of stochastic processes. In order to describe those processes in terms of distributional properties Fokker-Planck equations (FPEs) can be used. The relationship of stochastic processes via stochastic differential equations and FPEs is well known (cf. e.g. Gardiner (1997) or chapter 2 for a detailed discussion). In basic mathematical terms, the FPE is a differential equation. For jump-diffusion processes this can be either a partial differential equation or a delay differential equation. In both cases, FPEs describe the evolution of the probability density function of the random variable described by the stochastic differential equation.

We will focus on a model involving two Poisson processes that can be used to describe a simple labour market model with precautionary savings in spirit of Aiyagari (1994). This setup is used for example by Bayer and Wälde (2013) to show the existence and uniqueness of individual optimal behaviour together with a formal description of the wealth distribution by FPEs. Solving those equations is the next natural step for a better understanding of the dynamics of wealth and related concepts as for example poverty in a society described by such a framework. Nevertheless this is a non-trivial task and cannot be done analytically. Therefore, we will look into possible numerical solution methods.

Research question - Using FPEs to describe distributional properties leads to the problem of solving differential equations. Focusing on a basic continuous-time labour market model with precautionary savings, how can we solve the arising system of differential equations describing the probability density function of wealth? To answer this question, we present the so-called "method of characteristics" as a possible (numerical) solution technique and two finite-difference methods. Further, we compare those solution meth-

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ods for two versions of the model with different instantaneous utility function. Doing this, we discuss advantages and disadvantages of each method.

Setup - We focus on the individual level in a labour market with stochastic job separation and job finding. An individual faces constant payments while employed and in the case of a job loss constant unemployment benefits will be paid. The only possible way to insure against the loss in income during unemployment is the investment in a riskless asset. This setup is in-line with Aiyagari (1994) and in its continuous-time version was first used by Bayer and Wälde (2013). We will distinguish two cases of individual instantaneous utility: constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA). Both forms are widely used in economic literature (see discussion below), but the functional form does influence the complexity of the resulting FPEs. In both cases, we have to solve a coupled system of linear first-order partial differential equations in two independent variables. In the case of CARA utility, we get only constant coefficients allowing for a further analytical analysis, whereas the CRRA-case can only be analysed by numerical means.

In general, one possible solution method for this type of partial differential equations is the method of characteristics (MoC). This method reduces the system of partial differential equations into a family of ordinary differential equations. Those are, a priori, easier to solve. As known in literature (cf. e.g. DuChateau and Zachmann, 2002) the method of characteristics can provide an explicit solution for a homogeneous constant coefficient system of hyperbolic partial differential equations. In the case of inhomogeneous systems or systems without constant coefficients this method fails to provide an exact solution. Nevertheless, it is possible to apply this method numerically. Additionally, the new family of ordinary differential equations do reveal additional insights to the underlying model.

To provide alternatives to the MoC, we discuss the Lax-Friedrich method and the Crank-Nicolson method. Both are finite-difference methods, i.e. they approximate the partial derivatives involved by difference quotients of certain types and are by definition not able to provide an explicit solution.

Findings - As both our setups deal with hyperbolic PDEs that do not have an analytical closed form solution, we do rely on numerical methods. Both, the MoC and the finite difference methods used in this work are well known concepts in numerical mathematics, that are applied for this class of problem. Due to the additional insights provided by the method of characteristics, we prefer this approach to compute solutions in both setups. One direct observation that can be made using the MoC, is the influence of unemployed and employed individuals to the wealth distribution over time. Both groups of the labour market are responsible for the widening of the area of support of the wealth distribution, but the unemployed are responsible for a downward motion of the lower bound and equivalently the upper bound is influenced by employed individuals. This fact is not surprising, but we are able to explain this phenomena in more detail, compared to the other methods used.

Using the methods presented in this paper, we can compute the evolution of the wealth distribution at any point in time and especially for a large time-span. This presentation of a stable solution technique strengthens the importance of the FPEs as a powerful analysis tool for economic setups.

Literature review - This chapter is motivated by the analysis of FPEs in an economic context. Originating in mathematics and physics, there are many textbooks covering the theory of those equations[33]. Examples are for example Risken (1989), Gardiner (1997) or Stokey (2009). Chapter 2 of this thesis gives a detailed introduction to the concept of FPEs and how they can be used for a variety of economic models.

Focusing on the solution methods used in this chapter, they can also be found in a wide array of mathematical textbooks. One example is DuChateau and Zachmann (2002), who describe ways to model physical systems and at the same time give an introduction of numerical solution techniques. Abbott (1966) devotes a whole book on the MoC. Basically, most textbook in numerical mathematics dealing with partial differential equations, will at least include a short discussion on the method of characteristics and/or finite difference methods. Without any specific order and without being exhaustive, examples are Hanke-Bourgeois (2002), Larsson and Thomee (2003), Pinchover and Rubinstein (2005), or Quarteroni (2009).

Comparing the MoC and finite difference methods, "the major strength of the numerical method of characteristics [...[is]...] that it tracks information about the solution [...] along approximations to the characteristics." (DuChateau and Zachmann (2002), p. 449) Further the authors mention that the MoC is better capable in dealing with solutions containing sharp fronts. The disadvantage is the shift away from the original grid towards an irregularly spaced grid. Here lies the advantage of the finite-difference methods, for which the structure of the grid is never changed. Nevertheless we miss the information that the solution holds along characteristics (cf. DuChateau and Zachmann, 2002).

As a major part of the application is concerned with different types of utility functions, we want to motivate the choice of CARA and CRRA utility. Both types of utility functions are widely used in many fields of economic literature. Focusing on labour market literature, Shimer and Werning (2008) is one example where the influence of both CARA and CRRA utility is analysed in terms of optimal unemployment insurance. In line with our approach, Shimer and Werning use the CARA utility as benchmark for the more complicated CRRA case. Acemoglu and Shimer (1999, 2000) also analyse the effect of unemployment insurance in terms of labour productivity. In their earlier paper, they assume CARA utility leading to a closed form solution. Using CRRA utility in their later work, the authors rely on numerical computations. Other examples for CRRA utility in a search and matching framework are for example Krusell et al. (2010), Bils et al. (2009), or Hubbard et al. (1994, 1995), who analyse precautionary saving motives.

Outline - Section 5.2 establishes the economic framework for our analysis. We describe how the wealth distribution over time can be described by FPEs for both CARA and CRRA utility functions. Section 5.3 consists of two parts. We start with an overview over the MoC together with required numerical methods to solve the characteristic equations. The second part discusses finite difference methods and gives a short explanation of the Lax-Friedrich and the Crank-Nicolson methods. Section 5.4 uses those methods to solve the CARA case, before section 5.5 solves the CRRA case. As the CARA case is assumed to be the easier case, we will start here with an analytical solution attempt. For the CRRA case, we only give a numerical solution together with a comparative statics analysis. The latter showing the influence of the model parameters on the evolution of the wealth distribution. Finally, section 5.6 concludes.

[33] In some textbooks the term "forward Kolmogorov equation" is used rather then FPEs.

5.2 Economic framework

We will give a brief summary of the model, leading to the Fokker-Planck equations (FPEs), whose numerical solution is the main focus of this work. For a detailed explanation of the model we refer the reader to Bayer and Wälde (2013) [BW13 in what follows].

5.2.1 Wealth density over time

We start with a standard individual intertemporal maximization problem, i.e. an individual maximizes his/her individual intertemporal utility function given by

$$U(t) = E \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\, d\tau,$$

where in common notation r is the fixed interest rate, ρ is the time preference rate and instantaneous utility of consumption is denoted by $u(c(\tau))$. This maximization takes place in an uncertain economy that can be summarized by the following two restrictions: First, we have the standard budget constraint

$$da(t) = \{ra(t) + z(t) - c(a(t), z(t))\}\, dt, \tag{5.1}$$

where $z(t) \in \{w, b\}$ denotes the income. The income itself follows the stochastic differential equation

$$dz(t) = (w - b)\, dq_\mu(t) - (w - b)\, dq_s(t) \tag{5.2}$$

and is therefore the second constraint of the maximization. For both, eq. (5.1) and eq. (5.2), "w" denotes the wage and "b" represents unemployment benefits. As the income is linked to the employment status, we also think of "w" as indication for an employed individual and "b" indicates unemployment. Further, we assume that $w > b(\geq 0)$, i.e. employment is the preferred state as it yields a higher income. By $q_s(t)$ we denote the homogeneous Poisson process modelling job separation with arrival rate s. Job finding is described by the Poisson process $q_\mu(t)$ with arrival rate μ. Additionally, we assume that q_s and q_μ are mutually independent and that $\infty > \mu, s > 0$.

Let us give a more informal intuition of this model: As we have only two processes that model job transition and those two processes are assumed to be mutually independent, we do not allow for job-to-job transition. Individuals will either be unemployed and find a job with arrival rate μ or in case of a working individual, he/she keeps a job until the match is destroyed with arrival rate s. To explain eq. (5.2) in more detail, assume the case of an employed individual: Then (5.2) simplifies to $dz(t) = (-w + b)\, dq_s$ as a job offer will not arrive. Assume that at time $\tilde{t}$ a job loss takes place, i.e. $dq_s(\tilde{t}) = 1$. In that instance income and employment status changes by $(-w + b)$, i.e. the individual looses his/her wage&job and receives unemployment benefits instead ($z(\tilde{t}) = w + (-w + b) = b$). The income after $\tilde{t}$ will only change once $dq_\mu(t) = 1$, i.e. the individual finds another job. The only way an individual can self-insure against that reduction of income during unemployment is a riskless asset a. This asset changes according to the budget constraint (5.1) in two ways: Assets increase by the interest rate and the income but are reduced by consumption. Depending on the size of those factors overall behaviour of assets or wealth is controlled.

The influence of the stochastic income implies that assets, a, is a random variable, too. Hence, for each point in time we have to consider the discrete random variable z and the continuous random variable a (here and in the following derivation we suppress the time argument for readability). Going back to our research question, i.e. the distribution of wealth, we introduce $p(a,t)$ as the probability density function of wealth. This density can be written according to (cf. BW13, eq. (18)):

$$p\left(a,t\right)=p^{w}(a,t)+p^{b}\left(a,t\right), \tag{5.3}$$

where $p^{z}\left(.\right), z\in\{w,b\}$ will be called "sub-densities" and can be described as a product of a conditional density $p\left(a,t|z\right)$ with the probability of being in state z. Using eqs. (5.1)-(5.3) the FPEs can be computed following a standard derivation (cf. BW13 or chapter 2 of this thesis.). In matrix notation with vectors[34]

$$\hat{p}\left(a,t\right)=\begin{pmatrix}p^{w}(a,t)\\ p^{b}\left(a,t\right)\end{pmatrix} \text{ and } \hat{c}\left(a\right)=\begin{pmatrix}c^{w}(a)\\ c^{b}\left(a\right)\end{pmatrix}$$

the FPE reads

$$\frac{\partial}{\partial t}\hat{p}+B\frac{\partial}{\partial a}\hat{p}=C\hat{p}, \tag{5.4}$$

where

$$B=\begin{pmatrix}ra+w-c^{w}\left(a\right) & 0\\ 0 & ra+b-c^{b}\left(a\right)\end{pmatrix},$$

$$C=\begin{pmatrix}\frac{\partial}{\partial a}c^{w}\left(a\right)-r-s & \mu\\ s & \frac{\partial}{\partial a}c^{b}\left(a\right)-r-\mu\end{pmatrix},$$

and $\frac{\partial}{\partial t}\hat{p}=\left(\frac{\partial}{\partial t}p^{w}(a,t),\frac{\partial}{\partial t}p^{b}\left(a,t\right)\right)^{\mathrm{T}}$. Hence, we have to solve a coupled first order linear system of partial differential equations in two independent variables. The instantaneous utility function has not been used up to this point, but it can now be used to look closer at B and C or to be more precise at $\hat{c}\left(a\right)$ (this will be done in the following section).

Finally, it remains to be stated that in order to compute a unique solution to a first order PDE some initial conditions are required. Therefore we assume that values $\hat{p}\left(a,0\right)$ are known.

5.2.2 Two types of risk aversion

Any solution to eq. (5.4) must take optimal consumption into consideration. In order to determine this optimal policy rule, we need to use the maximized Bellman equation. This equation depends on the assumed form of instantaneous utility $u\left(\cdot\right)$. As discussed in the introduction, we focus on two types: constant absolute risk aversion (CARA) and constant relative risk aversion (CRRA). To ensure the existence of a unique solution for the optimal consumption path in the CRRA case we require one additional assumption, i.e. $0<r<\rho$ (cf. BW13). This assumption is assumed to hold for CARA utility as well.

[34] We introduce the abbreviation $c^{z}\left(a\right)$ for the function $c\left(a\left(t\right),z\left(t\right)\right), z\in\{w,b\}$ to reduce notation.

Constant absolute risk aversion

We start with a CARA utility function, i.e.

$$u(c) = -e^{-\gamma c},$$

where γ describes the risk aversion. In this case, we know that the optimal consumption of an individual is linear (cf. e.g. Shimer and Werning, 2007 & 2008). Assuming the functional form of

$$c^z(a) = ra + z + m_z, \;\; z \in \{w, b\} \tag{5.5}$$

with $m_w, m_b \in \mathbb{R}$ we can use the derivative of the maximized Bellman equation[35] to determine a non-linear system to determine those constants. This system reads

$$\begin{aligned} r\left[1 + m_w\gamma\right] - s - \rho + se^{\gamma[w-b+m_w-m_b]} &= 0, \\ r\left[1 + m_b\gamma\right] - \mu - \rho + \mu e^{-\gamma[w-b+m_w-m_b]} &= 0, \end{aligned} \tag{5.6}$$

and numerical computations have shown that, for realistic parameter values for r, γ, μ, s, w and b, the consumption specific parameters are $m_w < 0$ and $m_b > 0$.

Using optimal consumption (5.5) together with the budget constraint (5.1) shows that employed individuals increase their wealth. For unemployed individuals the opposite is true and individual assets will decrease. Hence, this optimal consumption can resemble a precautionary savings effect as desired by economic intuition.

Going back to the FPE (5.4), optimal consumption due to CARA preferences simplifies matrices B and C according to

$$B = \begin{pmatrix} -m_w & 0 \\ 0 & -m_b \end{pmatrix} \text{ and } C = \begin{pmatrix} -s & \mu \\ s & -\mu \end{pmatrix}.$$

This is still a coupled first order linear system of partial differential equations in two independent variables as C is a full matrix. Nevertheless, in comparison to the initial system, matrices B and C are constant, allowing a further analytical analysis (cf. section 5.4).

Constant relative risk aversion

Instantaneous utility reflects CRRA with risk aversion parameter σ, if

$$u(c) = \begin{cases} \frac{c^{1-\sigma}-1}{1-\sigma}, & \sigma > 0, \sigma \neq 1, \\ \ln(c), & \sigma = 1. \end{cases} \tag{5.7}$$

In the case of CRRA utility, optimal consumption paths for employed and unemployed individuals do not have a closed-form solution. As shown in chapter 4 of this thesis, optimal policy rules can be determined using a numerical solution to the Keynes-Ramsey rule. According to the numerical results as well as results proven in BW13, we know the following properties. First, optimal consumption is a concave function for both employment states. Second, for any given wealth level consumption of an employed individual is larger than consumption of an unemployed individual. Finally, the precautionary savings

[35] For a derivation of the maximized Bellman equation in this setup see BW13 or chapter 4 of this thesis.

motive does still occur as employed individuals will increase their wealth and unemployed individuals will decrease their wealth (given that we are in the interval $[-b/r, a_w^*]$ as described in chapter 4).

In terms of the FPE (5.4), we cannot simplify matrices B and C any further.

5.3 Numerical concepts

Having established the basic problem, this section gives a brief overview of possible numerical solution methods. Before we start with the actual methods, we introduce some notation. We will approximate the exact solutions of $p(a,t)$ for a two-dimensional grid. Therefore, we define for $a \in [aMin, aMax]$ and $t \in [0,T]$ the following entities[36]:

Let $\Delta t = [t_1, ..., t_n]$ be a equidistant grid of n knots where $t_1 = 0, t_n = T$. Additionally we define $h_t = t_{j+1} - t_j$. In a similar fashion we introduce an equidistant grid $\Delta a = [a_1, ..., a_m]$, with $h_a = a_{i+1} - a_i$ (where $a_1 = aMin$ and $a_m = aMax$). To shorten notation for any evaluation of the density and sub-densities of wealth, we also introduce $p_{i,j}^z = p^z(a_i(t_j), t_j)\,,\ z \in \{w, b\}\,.$

As a next step we introduce the MoC, followed by a short discussion of the Lax-Friedrich method and the Crank-Nicolson method. The latter two being examples of finite-difference methods. Finally, we conclude with a comparison of both methods.

5.3.1 The method of characteristics

Solving a PDE by the MoC involves locating a curve along which the solution follows an ODE. Such a curve is called "characteristic curve" or simply a "characteristic", what explains the name for the method itself. A detailed discussion of this method in the context of PDEs can be found in many mathematical textbooks as for example Abbott (1966), DuChateau and Zachmann (2002) or Larsson and Thomée (2003). The main advantage of the MoC is the knowledge of the characteristics and the hereby gained additional information, that can give additional insight in the economic mechanisms.

The following derivation is taken from Mattheij, Rienstra and ten Thije Boonkkamp (2005), ch. 2.2. As system (5.4) is already a very special case and as we would like to establish the MoC in more detail, we will give a derivation for the general equation

$$\mathbf{G}\frac{\partial \mathbf{p}}{\partial t} + \mathbf{H}\frac{\partial \mathbf{p}}{\partial a} = \mathbf{c}(\mathbf{a}, \mathbf{p})\,, \tag{5.8}$$

where $\mathbf{p}(x,t) = (p_1(x,t), p_2(x,t))^{\mathrm{T}}$ and $\mathbf{G}, \mathbf{H} \in \mathbb{R}^{2\times 2}$ are non-singular[37].

The first task is to "de-couple" the partial derivatives of p_1 and p_2, i.e. we try to transform matrices $\mathbf{G}$ and $\mathbf{H}$. Therefore, we need a transformation matrix $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\left(\mathbf{H}\mathbf{G}^{-1}\mathbf{S}\right) = \mathbf{\Lambda},$$

[36] In order to keep this introduction as general as possible we use $aMin, aMax$. Nevertheless, due to economic reasoning, we know that $aMin = -\frac{b}{r}$, the natural borrowing limit. Also $aMax$ is given by the endogeneous upper wealth limit a_w^* (cf. chapter 4).

[37] We choose a two dimensional system in accordance to our original problem. This method does hold for any dimension.

where $\mathbf{\Lambda}$ is a diagonal matrix with entries λ_1, λ_2. Given the existence of $\mathbf{S}$, we can define the characteristic variable as

$$\tilde{\mathbf{p}}(x,t) \equiv \mathbf{S}^{-1}\mathbf{G}\mathbf{p}(x,t).$$

Multiplying the initial PDE (5.8) by $\mathbf{S}^{-1}$ and using the characteristic variable together with $\mathbf{\Lambda}$ we end up with the the de-coupled system

$$\frac{\partial \tilde{\mathbf{p}}}{\partial t} + \mathbf{\Lambda}\frac{\partial \tilde{\mathbf{p}}}{\partial x} = \tilde{\mathbf{c}} = \mathbf{S}^{-1}\mathbf{c}.$$

Each row of this matrix equation induces a characteristic corresponding to the the eigenvalue and eigenvector of $\mathbf{H}\mathbf{G}^{-1}$. The characteristic equations read

$$\frac{da}{dt} = \lambda_k,\ \frac{d\tilde{p}_k}{dt} = \tilde{c}_k \qquad (k = 1, 2), \tag{5.9}$$

where the solution of $\frac{d\tilde{p}_k}{dt} = \tilde{c}_k$ holds along the solution of $\frac{da}{dt} = \lambda_k$. To solve those equations we need additional information, i.e. boundary and/or initial values depending on the definition of a.

A first example

To demonstrate the MoC we will solve the following initial value problem:

$$\frac{\partial}{\partial t}p(a,t) + \alpha\frac{\partial}{\partial a}p(a,t) = 0,\ a \in \mathbb{R}, t \geq 0, \alpha \in \mathbb{N},$$

$$p(a,0) = e^a,\ \forall a \in \mathbb{R},$$

which is an advection equation describing a wave propagation.

Following the above derivation we end up with the characteristic equations

$$\frac{da}{dt} = \alpha, \tag{5.10a}$$

$$\frac{dp}{dt} = 0. \tag{5.10b}$$

How can we use those equations to obtain a solution to our initial problem? From eq. (5.10b) we see that the solution $p(a,t)$ is constant along $a = \alpha t + a_0$ ($a_0 = const$), which is the solution of the characteristic (5.10a) Using the initial condition yields that the solution to the initial problem is given by

$$p(a,t) = e^{a-\alpha t},$$

as for $t = 0$ the characteristic intersects the x-axis at a_0.

The solution can easily be verified: Using the chain rule yields

$$\frac{\partial}{\partial t}p(a,t) = -\alpha e^{a-\alpha t},$$

$$\frac{\partial}{\partial a}p(a,t) = e^{a-\alpha t},$$

and hence we get $-\alpha e^{a-\alpha t} + \alpha \cdot e^{a-\alpha t} = 0$ as required. Also the initial condition is fulfilled as $p(a,0) = e^{a-\alpha\cdot 0} = e^a$.

Solving the MoC-system

Looking back at the first example, we could compute a closed-form solution to the advection equation. Depending on the form of the characteristic equations this is not possible in every case, as it is not always possible to find closed form solutions for the ODEs given in eq. (5.9). Especially in our case of CRRA utility we will rely on solving those equations by numerical methods. Dealing with ODEs, we can access a broad range of numerical methods to solve such a system. As our analysis has shown, we can use fairly simple methods from the Runge-Kutta family to solve the characteristic equations. This section gives a brief overview of two methods that will be used in our analysis in sections 5.4 and 5.5.

We start with the explicit Euler method, which can be described as the easiest method in terms of computational effort. But as this method has shortcomings in terms of accuracy we present *the* Runge-Kutta method or classical Runge-Kutta method, which is also an explicit method but of fourth-order[38]. For a detailed description of the following solution techniques we refer the reader to any preferred textbook on numerical methods for ODEs as for example Hanke-Bourgeois (2002) or Judd (1998).

The explicit Euler method (eEuler)
To explain this method, we assume that we want to solve the general initial value problem $y' = f(t, y),\ y(0) = y_0$. The basic idea of the eEuler is to use the slope given by the ODE, i.e. $f(t, y)$, to compute the value of the function at a later point in time. This method relies on the assumption, that after a short time-interval the tangent is still a good approximation of the original curve. Hence, we start in y_0 and move forward with the initial slope $f(0, y_0)$. Computation of the new value involves only the information of the previous step, hence this method is called explicit.

In mathematical notation one step of the eEuler is given by

$$y(t_{j+1}) = y\left(t_j\right) + h_t f\left(t_j, y\left(t_j\right)\right)$$

(cf. Hanke-Bourgeois, 2002).

The **Runge-Kutta method (RK4):**
Additionally, we will use *the* Runge-Kutta method, which is also known as "classical Runge-Kutta method" or "RK4". In comparison to eEuler, where only the slope of the current time $f\left(t_j, y\left(t_j\right)\right)$ is used, RK4 uses a weighted average of four different values given by the direction field in the interval $[t_j, t_{j+1}]$. One step of RK4 is defined as.

$$y(t_{j+1}) = y\left(t_j\right) + h_t\left[\frac{1}{6}k_1 + \frac{1}{3}k_2 + \frac{1}{3}k_3\frac{1}{6}k_4\right],$$

where $k_1 = f\left(t_j, y\left(t_j\right)\right)$. k_2 is the slope computed with half an explicit Euler step using k_1. In order to compute k_3, we perform half an explicit Euler step using k_2. The slope determined with that step is k_3. Finally, the slope computed with a whole explicit Euler step using k_3 is defined as k_4 (cf. Hanke-Bourgeois, 2002 - ex. 76.7).

This method is a special case of the family of Runge-Kutta methods defined as

$$y_{i+1} = y_i + h\sum_{j=1}^{s} b_j f(t_i + c_j h, \eta_j),$$

[38]Loosely speaking, a method is called of n-th order if the error term is $\mathcal{O}\left(h^{n+1}\right)$, where h is the distance between two grid points (cf. Hanke-Bourgeois, 2002).

with an appropriate choice of knots c_j, weights b_j and the number of stages s (cf. Hanke-Bourgeois, 2002)[39].

Of course, we could also use built-in methods of a numerical computing environment. As all computations are performed using Matlab[40], one available routine is for example *"ode45"*. This method is "based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair" (cf. Matlab - HelpFile). Using this function has a major drawback as we noticed in the process of testing our solution method. The routine *ode45* is designed to compute the solution of an ODE system in a large time interval. In our case, we want to solve the ODE just for a small period of time (cf. section 5.4). This leads to a huge amount of function calls, as we need to use *ode45* at each t_j and this has to be done for each point in Δa. Consequently, using *ode45* or any other built-in method needs a high amount of computation time once h_t and h_a is reasonable small. Nevertheless, *ode45* is of higher order as eEuler and RK4 and could result in a more accurate solution for some examples.

5.3.2 Finite difference methods as alternative concepts

In order to demonstrate possible alternatives, this section discusses two finite-difference methods. The basic idea for this type of numerical solution is to approximate the partial derivatives occurring in eq. (5.4) using difference quotients. Using for example the partial derivative with respect to wealth a, such a quotient reads

$$\frac{\partial}{\partial a} p\,(a,t) = \lim_{h \to 0} \frac{p(a+h,t) - p\,(a,t)}{h}.$$

Once we do not take the limit but the value of the difference quotient for a fixed h, we can use this as an approximation for the partial derivative itself. The so-called truncation error, i.e. the error we inevitably make by using an approximation instead of the exact value, can be determined using Taylor's theorem. Taking the Taylor approximation of a real valued function $f(x)$ around some point a, we know that $f(x) = \sum_n f^{(n)}(a) \frac{(x-a)^n}{n!}$. Hence the first-order Taylor approximation gives an approximation of the first derivative, i.e. $f'(a) \approx \frac{f(x)-f(a)}{(x-a)}$ and the truncation error can be determined using the higher order terms.

To determine a unique solution in a finite space we need initial as well as boundary conditions, i.e.

$$\begin{aligned} p^z(a_1,t) &= c_1^z(t), \\ p^z(a_m,t) &= c_m^z(t), \\ p^z(a,0) &= c_0^z(a), \end{aligned} \tag{5.11}$$

with $c_1^z, c_m^z, c_0^z \in C^\infty(\mathbb{R})$, $z \in \{w,b\}$. In our formulation of the original problem, we state only initial conditions. The additional boundary conditions are sometimes called "numerical boundary conditions", as those are only required to determine a unique solution with the help of the numerical finite difference methods (cf. Strikwerda (2004), p.85).

[39]Taking this definition, eEuler is also a Runge-Kutta methods. In fact, it is the simplest method of order 1.

[40]*MATLAB v.7.12.0.635 (R2011a), 32-bit*, The MathWorks, Inc., Natick, Massachusetts, United States.

Lax-Friedrich method

The Lax-Friedrich method applies the "forward in time, centered in space (FTCS)"-method to approximate the partial derivatives (cf. DuChateau and Zachmann, 2002). The approximations used are

$$\frac{\partial}{\partial t} p_{i,j+1}^z = \frac{p_{i,j+1}^z - \frac{1}{2}\left(p_{i-1,j}^z + p_{i+1,j}^z\right)}{h_t}, \tag{5.12a}$$

$$\frac{\partial}{\partial a} p_{i,j+1}^z = \frac{1}{2h_a}\left(p_{i+1,j}^z - p_{i-1,j}^z\right), \tag{5.12b}$$

with $z \in \{w, b\}$, $i = 2, ..., m-1$ and $j = 1, ..., n-1$.

How can this approximation be used for our problem? Suppose we have finished our computation of $p^w(.)$ and $p^b(.)$ for time t_j for all points in Δa and we want to determine values for the grid point a_i at the next point in time t_{j+1}. Starting with the first row of the matrix equation eq. (5.4) we have

$$\frac{\partial}{\partial t} p_{i,j+1}^w + B_{11} \frac{\partial}{\partial a} p_{i,j+1}^w = C_{11} p_{i,j+1}^w + C_{12} p_{i,j+1}^b,$$

where we used the definitions $B := \{B_{kl}\}, C := \{C_{kl}\}$. Using eq. (5.12) yields

$$p_{i,j+1}^w - \frac{1}{2}\left(p_{i+1,j}^w + p_{i-1,j}^w\right) + B_{11}\frac{h_t}{2h_a}\left(p_{i+1,j}^w - p_{i-1,j}^w\right) = h_t\left(C_{11} p_{i,j+1}^w + C_{12} p_{i,j+1}^b\right).$$

Rewriting this in terms of $p_{i,j+1}^z$, the values we want to compute, we get

$$\begin{pmatrix} 1 - h_t C_{11} & -h_t C_{12} \end{pmatrix} \begin{pmatrix} p_{i,j+1}^w \\ p_{i,j+1}^b \end{pmatrix} = \frac{1}{2}\left(p_{i+1,j}^w + p_{i-1,j}^w\right) - B_{11}\frac{h_t}{2h_a}\left(p_{i+1,j}^w - p_{i-1,j}^w\right).$$

Here all values occurring on the right hand side are known (as all values needed are evaluations at time t_j). The same holds if we look at the second equation of eq. (5.4). In summary, we get

$$\begin{pmatrix} 1 - h_t C_{11} & -h_t C_{12} \\ -h_t C_{21} & 1 - h_t C_{22} \end{pmatrix} \begin{pmatrix} p_{i,j+1}^w \\ p_{i,j+1}^b \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(p_{i+1,j}^w + p_{i-1,j}^w\right) - B_{11}\frac{h_t}{2h_a}\left(p_{i+1,j}^w - p_{i-1,j}^w\right) \\ \frac{1}{2}\left(p_{i+1,j}^b + p_{i-1,j}^b\right) - B_{22}\frac{h_t}{2h_a}\left(p_{i+1,j}^b - p_{i-1,j}^b\right) \end{pmatrix}.$$

This is a linear system of two equations for two unknowns $p_{i,j+1}^w$, $p_{i,j+1}^b$), yielding one unique solution if and only if $1 - h_t\left(C_{11} - C_{22}\right) + h_t^2\left(C_{11}C_{22} - C_{12}C_{21}\right) \neq 0$.

Here, we have to solve a linear system of equations for $p_{i,j+1}^z$ instead of having a explicit formula for those variables. This is due to the fact that we deal with a coupled system.

In total, we have to solve $m-2$ such systems, i.e. for each grid point of Δa, in order to proceed from t_j to t_{j+1}. As those systems are only two dimensional this is a feasible task and slows the computation in an acceptable way[41]. Note that due to the FTCS we can compute only values for the interval $[a_2, a_{m-1}]$. Values for a_1 and a_m have to be taken from the boundary conditions $c_1^z(t)$ and $c_m^z(t)$.

Concerning the stability of this method, we know a necessary condition, i.e. the Courant-Friedrich-Levy (CFL) condition. Basically, the CFL condition gives a restriction on the

[41] Tests showed that solving $(m-2)$-times a two-dimensional systems is faster than solving a $(2m-4)$-dimensional system once.

relation of h_a and h_t. For a system with constant coefficients (i.e. matrix B in eq. (5.4) needs to be constant) the CFL condition is

$$\left|\lambda_i \frac{h_t}{h_a}\right| \leq 1,$$

where λ_i are the (real) Eigenvalues of matrix B (cf. DuChateau and Zachmann, 2002). Note that in the case of a diagonal matrix $\lambda_i \equiv B_{ii}$.

Crank-Nicolson method

This is an example of an implicit method and hence is unconditionally stable. Furthermore the local truncation error is in the order of $\mathcal{O}(h_t^2 + h_a^2)$ (cf. DuChateau and Zachmann (2002), table 9.2.2).

The used approximations are (for $z \in \{w, b\}$)

$$\begin{aligned}
\frac{\partial}{\partial a} p^z_{i,j+1} &= \frac{1}{2}\left[\frac{p^z_{i+1,j+1} - p^z_{i-1,j+1}}{2h_a} + \frac{p^z_{i+1,j} - p^z_{i-1,j}}{2h_a}\right] \\
&= \frac{1}{4h_a}\left[p^z_{i+1,j+1} - p^z_{i-1,j+1} + p^z_{i+1,j} - p^z_{i-1,j}\right], \\
\frac{\partial}{\partial t} p^z_{i,j+1} &= \frac{p^z_{i,j+1} - p^z_{i,j}}{h_t}, \\
p^z_{i,j+1} &= \frac{1}{2}\left(p^z_{i,j+1} + p^z_{i,j}\right).
\end{aligned}$$

To demonstrate the application of the Crank-Nicolson method, we assume that we know the values of $p^w_{i,j}$, $p^b_{i,j}$ $\forall i$. The next step is the computation of values $p_{i,j+1}$, $\forall i$. Introducing a vector containing the unknowns, i.e.

$$\hat{p}_{i,j+1} := \left[p^w_{2,j+1}, ..., p^w_{m-1,j+1}, p^b_{2,j+1}, ..., p^b_{m-1,j+1}\right],$$

we end up with the following system that needs to be solved:

$$L \cdot \hat{p}_{i,j+1} = z.$$

Here $L \in \mathbb{R}^{2(m-2)\times 2(m-2)}$ is a band matrix[42] allowing to use optimized algorithms for spare matrices.

5.3.3 Summary of numerical methods

Having introduced both the MoC and two finite difference methods, it becomes obvious that the MoC has some appealing advantages. Firstly, the MoC is not a pure numerical method as it can be used to compute a closed form solution in some cases. If this is not possible, we can still use the characteristic equations together with numerical solution methods for ODEs to determine a numerical solution. Here, basic methods that are either readily available or that can easily be implemented are suitable to solve the occurring ODEs. Finite difference methods are by definition numerical methods and can

[42]For an explicit formula of L and z and a detailed derivation we refer the reader to appendix 5.7.2.

never lead to a closed form solution. From our point of view the complexity in terms of computational and implementation effort is comparable for both solution methods and does not favour any of the methods.

The final advantage of the MoC is the additional insight, that can be gained from the characteristic equations itself. For our two examples we are able to explain movements in the wealth distribution due to the precautionary savings motive as will be shown in the next two sections.

In summary, the MoC is a promising method for the analysis of distributional properties as described by FPEs and hence will be discussed in more detail in the next two sections for our model with CARA and CRRA utility.

5.4 Application I - CARA

The FPEs incorporating CARA utility are

$$\mathrm{PDE}^w \equiv \frac{\partial}{\partial t} p^w(a,t) - m_w \frac{\partial}{\partial a} p^w(a,t) + s p^w(a,t) - \mu p^b(a,t) = 0, \tag{5.13a}$$

$$\mathrm{PDE}^b \equiv \frac{\partial}{\partial t} p^b(a,t) - m_b \frac{\partial}{\partial a} p^b(a,t) + \mu p^b(a,t) - s p^w(a,t) = 0, \tag{5.13b}$$

where parameters m_w, m_b, s and μ are constant. Additionally, we assume that some initial distribution $\hat{p}(a,0) = \left(p^w(a,0), p^b(a,0)\right)^{\mathrm{T}}$ $\forall a$ is known.

5.4.1 The characteristic equations

Following section 5.3.1, we obtain the characteristic equations (cf. eq. (5.9)):

$$\frac{da_{C_1}}{dt} = -m_w, \tag{5.14a}$$

$$\frac{da_{C2}}{dt} = -m_b, \tag{5.14b}$$

$$\frac{d\tilde{p}^w}{dt} = -s\tilde{p}^w + \mu\tilde{p}^b, \tag{5.14c}$$

$$\frac{d\tilde{p}^b}{dt} = s\tilde{p}^w - \mu\tilde{p}^b. \tag{5.14d}$$

It is important to note that eq. (5.14c) holds on the characteristic line (5.14a) only and that eq. (5.14d) holds on the characteristic line (5.14b). In a slight abuse of notation, the indices of a_{C_1} and a_{C_2} indicate the fact that those equations describe the characteristics and not the grid points of Δa.

5.4.2 The analytical solution of the ODE system

This section will show that it is not possible to find a closed form solution to the FPEs even for this "easy" case of CARA utility, i.e. constant coefficients.

In order to get further insights into the properties of our solution, we now solve for the characteristic paths. Solving the ODEs in eq. (5.14a) and eq. (5.14b) trivially gives

$$a_{C_1}(t) = a_{C_1}(0) - m_w t, \tag{5.15}$$

$$a_{C_2}(t) = a_{C_2}(0) - m_b t. \tag{5.16}$$

Starting with the solution to the inhomogeneous ODE (5.14c) with $\tilde{p}^w(0)$ as initial value, the solution reads

$$\tilde{p}^w(t) = e^{-st}\tilde{p}^w(0) + \mu \int_0^t e^{-s[t-\tau]}\tilde{p}^b(\tau)\,d\tau.$$

Using the definition of $\tilde{p}^z(t) = p^z(a_{C_1}(t), t),\ z \in \{w, b\}$ we can rewrite this as

$$p^w(a_{C_1 1}(t), t) = e^{-st}p^w(a_{C_1}(0), 0) + \mu \int_0^t e^{-s[t-\tau]}p^b(a_{C_1}(\tau), \tau)\,d\tau.$$

As this solution holds on the characteristic line (5.15) only, we need to substitute in an appropriate way, i.e. by using

$$a_{C_1}(0) = a_{C_1}(t) + m_w t,$$

$$a_{C_1}(\tau) = a_{C_1}(0) - m_w\tau = a_{C_1}(t) - m_w[\tau - t].$$

Finally, we obtain that

$$p^w(a_{C_1 1}(t), t) = e^{-st}p^w(a_{C_1}(t) + m_w t, 0) + \mu \int_0^t e^{-s[t-\tau]}p^b(a_{C_1}(t) - m_w[\tau - t], \tau)\,d\tau.$$

Dropping the "$_{C_1}(t)$"-argument after a yields

$$p^w(a, t) = e^{-st}p^w(a + m_w t, 0) + \mu \int_0^t e^{-s[t-\tau]}p^b(a - m_w[\tau - t], \tau)\,d\tau. \tag{5.17}$$

In other words, we have integrated eq. (5.14c) along the characteristic (5.14a).

We can treat $\tilde{p}^b(t)$ in a similar fashion. This time we have to consider, that eq. (5.14b) is the characteristic corresponding to eq. (5.14d). Using the same algebraic steps as above, we end up with

$$p^b(a, t) = e^{-\mu t}p^b(a + m_b t, 0) + s \int_0^t e^{-\mu[t-\tau]}p^w(a - m_b[\tau - t], \tau)d\tau. \tag{5.18}$$

To sum this up, the solution to the system (5.13) is given by the system of integral eqs. (5.17) and (5.18). Looking at the first terms in (5.17) and (5.18) we can see that initial functions $p^z(\cdot, 0), z \in \{w, b\}$ are "propagated trough time" in two ways: Firstly, they change shape (but keep of course their functional form) as the $m_w t$ and the $m_b t$ terms are visible in $p^z(\cdot, 0), z \in \{w, b\}$. Secondly, they "lose importance" as s and μ are positive and the exponential factor vanishes over time.

Are (5.17) and (5.18) solutions to the PDE system?

The key ingredients to answer this question are the partial derivatives of p^w and p^b, i.e.

$$\frac{\partial}{\partial a}p^w(a,t) = e^{-st}\frac{\partial p^w}{\partial a}(a+m_w t,0) + \mu\int_0^t e^{-s[t-\tau]}\frac{\partial p^b}{\partial a}\left(a-m_w\left[\tau-t\right],\tau\right)d\tau$$

and[43]

$$\begin{aligned}\frac{\partial}{\partial t}p^w(a,t) =& e^{-st}\frac{\partial p^w}{\partial a}(a+m_w t,0)m_w - se^{-st}p^w(x+m_w t,0)\\ &+\mu\frac{\partial}{\partial t}\int_0^t e^{-s[t-\tau]}p^b\left(a-m_w\left[\tau-t\right],\tau\right)d\tau\\ =& e^{-st}\left(\frac{\partial p^w}{\partial t}(a+m_w t,0) - sp^w(a+m_w t,0)\right) + \mu p^b(a,t)\\ &+\mu\int_0^t e^{-s[t-\tau]}\left(-sp^b\left(a-m_w\left[\tau-t\right],\tau\right) + m_w\frac{\partial p^b}{\partial t}\left(a-m_w\left[\tau-t\right],\tau\right)\right)d\tau.\end{aligned}$$

For p^b we get similar results:

$$\begin{aligned}\frac{\partial}{\partial a}p^b(a,t) =& e^{-\mu t}\frac{\partial p^b}{\partial a}(a+m_b t,0) + s\int_0^t e^{-\mu[t-\tau]}\frac{\partial p^w}{\partial a}\left(a-m_b\left[\tau-t\right],\tau\right)d\tau,\\ \frac{\partial}{\partial t}p^b(a,t) =& e^{-\mu t}\left(\frac{\partial p^b}{\partial t}(a+m_b t,0) - \mu p^b(a+m_b t,0)\right) + sp^w(a,t)\\ &+s\int_0^t e^{-\mu[t-\tau]}\left(-\mu p^w\left(a-m_b\left[\tau-t\right],\tau\right) + m_b\frac{\partial p^w}{\partial t}\left(a-m_b\left[\tau-t\right],\tau\right)\right)d\tau.\end{aligned}$$

Now we are able to verify that integral equations (5.17) and (5.18) are indeed solutions to (5.13a) and (5.13b). Demonstrating this exemplary for PDE^w, we look at the following factors in turns:

We begin with the examination of $\frac{\partial}{\partial t}p^w(a,t) - \mu p^b(a,t)$, which simplifies to

$$\begin{aligned}\frac{\partial}{\partial t}p^w(a,t) - \mu p^b(a,t) =& e^{-st}\left(\frac{\partial p^w}{\partial a}(a+m_w t,0)m_w - sp^w(a+m_w t,0)\right)\\ &- s\mu\int_0^t e^{-s[t-\tau]}p^b\left(a-m_w\left[\tau-t\right],\tau\right)d\tau\\ &+ m_w\mu\int_0^t e^{-s[t-\tau]}\frac{\partial p^b}{\partial t}\left(a-m_w\left[\tau-t\right],\tau\right)d\tau.\end{aligned}$$

The remaining two factors incorporating our integral expression for p^w and p^b are

$$\begin{aligned}&-m_w\frac{\partial}{\partial a}p^w(a,t) =\\ &\qquad -m_w e^{-st}\frac{\partial p^w}{\partial a}(a+m_w t,0) - m_w\mu\int_0^t e^{-s[t-\tau]}\frac{\partial p^b}{\partial a}\left(a-m_w\left[\tau-t\right],\tau\right)d\tau,\\ &\qquad sp^w(a,t) = se^{-st}p^w(a+m_w t,0) + s\mu\int_0^t e^{-s[t-\tau]}p^b\left(a-m_w\left[\tau-t\right],\tau\right)d\tau.\end{aligned}$$

[43]We use the chain rule to obtain: $\frac{\partial}{\partial t}\left(p^w(a+m_w t,0)\right) = \frac{\partial p^w}{\partial a}(a+m_w t,0)m_w + \frac{\partial p^w}{\partial t}(a+m_w t,0)\cdot 0$

Building the sum over those three expression, eliminates all terms and we have proven (5.13a). In order to proof that (5.13b) holds with (5.17) and (5.18), we can follow the same line of argument.

Solving the system of integral equations (5.17) & (5.18)

As our goal is to find a closed form solution for $\hat{p}$, we need to examine the system of integral equations (5.17) and (5.18) closer. The limits of the integrals involved consists of one fixed value 0 and the upper limit is t. Hence we can classify our system as a Volterra-type equation of second-order.

It is well documented, that some classes of integral equations of the Volterra-type can be solved using differentiation. According to Bronstein et al. (2001), ch. 11.4.2 this method works especially in cases of a polynomial kernel. In our case we have an exponential kernel; nevertheless, we will try the "derivation trick" anyhow.

Looking at the time derivative (as computed previously) shows that

$$
\begin{aligned}
\frac{\partial}{\partial t} p^w(a,t) &= -se^{-st} p^w(a + m_w t, 0) + e^{-st} \frac{\partial p^w}{\partial t}(a + m_w t, 0) + \mu p^b(x,t) \\
&\quad + \mu \int_0^t e^{-s[t-\tau]} \left(-sp^b\left(a - m_w\left[\tau - t\right], \tau\right) + m_w \frac{\partial p^b}{\partial t}\left(a - m_w\left[\tau - t\right], \tau\right) \right) d\tau \\
&= -s \left[e^{-st} p^w(x + m_w t, 0) + \mu \int_0^t e^{-s[t-\tau]} p^b\left(x - m_w\left[\tau - t\right], \tau\right) d\tau \right] \\
&\quad + e^{-st} \frac{\partial p^w}{\partial t}(a + m_w t, 0) + \mu p^b(a,t) \\
&\quad + m_w \mu \int_0^t e^{-s[t-\tau]} \frac{\partial p^b}{\partial t}\left(a - m_w\left[\tau - t\right], \tau\right) d\tau.
\end{aligned}
$$

Substituting the solution back in PDE^w yields

$$
\begin{aligned}
\frac{\partial}{\partial t} p^w(a,t) = &- sp^w(a,t) + e^{-st} \frac{\partial p^w}{\partial t}(a + m_w t, 0) + \mu p^b(x,t) \\
&+ m_w \mu \int_0^t e^{-s[t-\tau]} \frac{\partial p^b}{\partial t}\left(a - m_w\left[\tau - t\right], \tau\right) d\tau.
\end{aligned}
$$

Indeed, the simplification by derivation does not work in the our case. Hence we stop here with the analytical analysis and switch to numerical computation methods.

5.4.3 The numerical solution

In the case of CARA utility, we know that $m_w < 0$ and $m_b > 0$ (cf. section 5.2.2). We have already shown that the characteristics are straight lines. Those two facts yield that once we focus on our grid Δa, we have two movements: The solution of eq. (5.14c) is valid along a line with positive slope, i.e. they "move" from left to right. At the same time, the solution to eq. (5.14d) is valid along a line with negative slope (cf. figure 5.1).

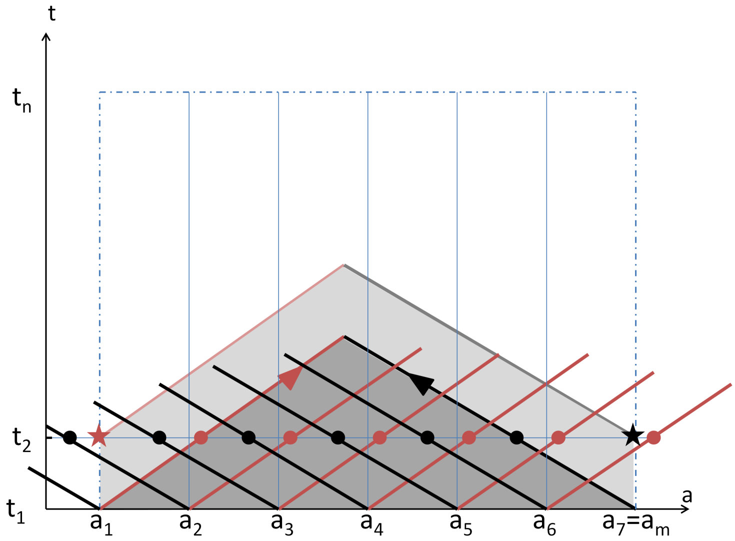

Figure 5.1: Sketch of the area of interest, showing schematic characteristics - red indicates characteristics linked to the state $z = w$, i.e. eq. (5.14a), and black lines are schematic drawings of the solution to eq. (5.14b).

As said before, the idea of the MoC is that the solution of eq. (5.21c) holds along eq. (5.21a). With reference to figure 5.1 we need to think of a third dimension containing this solution. For example, suppose we start at $p^w(a_i, t_1)$. For our chosen time step h_t, the solution of eq. (5.14c) would hold "above" the black dot on the corresponding black line through a_i. Hence we depart from our numerical grid points (a_i, t_j). In order to get values of the solutions at the original grid points, we have to apply for example interpolation methods.

As indicated in figure 5.1, we can interpolate the solution of p^w in the interval $I_w = [a_2 + \delta, a_7 + \varepsilon]$, $\delta, \varepsilon > 0$ and p^b can be interpolated in $I_b = [a_1 - \alpha, a_6 - \beta]$, $\alpha, \beta > 0$. As we need information of both p^w and p^b to solve the system, we end up with a decreasing area, i.e. $I_w \cap I_b \equiv [a_3, a_5]$ in our example drawn[44]. This large decrease is a result of the coarse grid. In general, the area in dark grey shows the largest possible area where we can solve our system relying on initial conditions only. One way to include the whole interval $[a_1, a_m]$ at t_2 would be extrapolation, but this is highly unreliable in terms of required properties on $\hat{p}$ as for example mass conservation.

A better way to solve this problem is to introduce boundary conditions, i.e.

$$p^w(a_1, t) = c_1^w(t), \tag{5.19a}$$

$$p^b(a_m, t) = c_m^b(t). \tag{5.19b}$$

For t_2 those boundary conditions are indicated as a star in figure 5.1. Please note that in order to end up with a well-posed problem, we are only allowed to define boundary conditions at the edge where the corresponding characteristic enters the finite interval (cf. Strikwerda, 2004)[45]. Once we have introduced those boundary conditions we can

[44]The true intersection set $I_w \cap I_b$ is larger, i.e. $[a_2 + \delta, a_6 - \beta]$, but as we rely on the numerical grid we have to choose grid points lying in the intersection set and hence we end up with $[a_3, a_5]$ as written above.

[45]A problem with initial as well as boundary conditions is called "initial-boundary value problem" in

interpolate the solutions for p^w and p^b for the whole Interval $[a_1, a_m]$. In order to compute values at t_3, we use the values computed for t_2 as initial conditions and proceed as in the first step. Repeating this procedure allows us to compute a solution for the complete time interval[46].

Following this discussion our generic algorithm[47] to compute a solution to the MoC at time t_{j+1} starting in t_j is:

Algorithm 5.1

1. *Solve the system of ODEs (5.14c) and (5.14d) for the interval* $[t_j, t_{j+1}]$, *for each* a_i.

2. *For each* a_i, *solve the characteristics (5.14a) and (5.14b), leading to a shifted grid* $\Delta a'$.

3. *As the solutions found in step 1 are valid at* $\Delta a'$, *use a suitable interpolation method to compute solutions at* (a_i, t_{j+1}).

Finally, we will present a first numerical solution to eq. (5.13) computed according to algorithm 5.1. The results shown in figure 5.2 were computed with the following parameter values:

$$\gamma = 2,\ r = 0.016, \rho = 0.02,\ w = 3.99,\ b = w/2,\ s = 0.67 \text{ and } \mu = 0.067.$$

Those parameter values imply that $m_w = -0.078$ and $m_b = 1.874$ according to eq. (5.6). Without the economic background those values are somehow arbitrary but satisfy all requirements. Nevertheless, we have chosen those values corresponding to plausible economic values[48]. The value of r corresponds to an annual interest rate of 5%. The arrival rates imply a job-duration of on average five years and an average unemployment spell of six months. The value of w implies a monthly wage of 1000$.

Finally, we assume initial distributions for $p^w(a, t_0), p^b(a, t_0) \sim N(10, 2)$, where we need to introduce an adjustment factor to ensure that $\int p(a, t_0)\, da = 1$. Therefore, we use a weighted sum, i.e.

$$p(a, t_0) = (1 - \omega_b) p^w(a, t_0) + \omega_b p^b(a, t_0),$$

where $\omega_b \in [0, 1]$ can be interpreted as the initial unemployment rate. For the solution shown in figure 5.2, we set $\omega_b = 0.2$.

Discussing the results shown in figure 5.2, we want to stress that panel B has a different scale. Panel C shows the advantage of a continuous-time model as we are able to show the surface of the density over the whole period. Panel A and B show only parts of the

mathematical literature. Strikwerda (2004) dedicates a complete chapter to methods on how to check if such problems are well-posed.

[46] Introducing interpolation as an additional numerical method, introduces another possible source of error. We will use *interp1*, a built-in Matlab function capable of cubic-spline interpolation. Assuming that we compute a smooth density as required by the economic motivation, this error should be negligible.

[47] For the application on a two-dimensional uncoupled system see appendix 5.7.1.

[48] As the model is in continuous-time, the parameter values must reflect this aspect. Hence we cannot use for example monthly or yearly data. This aspect is discussed in more detail in chapter 4 of this thesis.

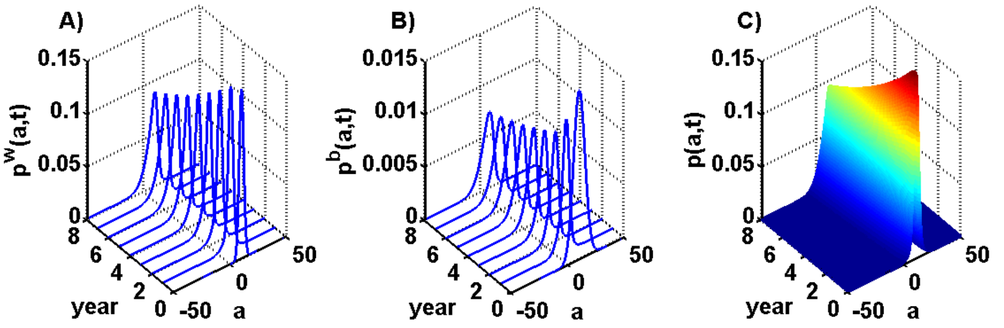

Figure 5.2: Numerical solution of eq. (5.13) computed for a time span of eight years. Panel A and B show sub-densities $p^w(a,t), p^b(a,t)$ for beginning of each year only. Panel C shows unconditional density $p(a,t)$.

continuous solution for the sub-densities, i.e. the result corresponding to the beginning of each year.

Looking at panel C, we can observe a decline of the absolute value of the mode. Hence the distribution needs to get broader over time, to ensure that $\int p(a,t)\, da = 1 \ \forall t$[49]. The decline of the mode as well as a wider area of support does also hold for $p^w(a,t)$ and $p^b(a,t)$ individually. Due to the assumption that wealth is similarly distributed for employed and unemployed individuals, the upward and downward drift is hard to see from figure 5.2. Nevertheless the values for m_w and m_b together with the budget constraint do explain that employed individuals increase their wealth, i.e. the upper bound of the support needs to increase as in the beginning 80% are employed. At the same time 20% are assumed to be unemployed and hence decrease their wealth, resulting in a decrease of the lower bound of support of $p(a,t)$.

5.5 Application II - CRRA

For the case of CRRA utility, the FPEs read

$$\frac{\partial}{\partial t}p^w(a,t) + [ra + w - c(a,w)]\frac{\partial}{\partial a}p^w(a,t) = \left[\frac{\partial}{\partial a}c(a,w) - r - s\right]p^w(a,t) + \mu p^b(a,t), \tag{5.20a}$$

$$\frac{\partial}{\partial t}p^b(a,t) + [ra + b - c(a,b)]\frac{\partial}{\partial a}p^b(a,t) = sp^w(a,t) + \left[\frac{\partial}{\partial a}c(a,b) - r - \mu\right]p^b(a,t). \tag{5.20b}$$

Following section 5.3.1, the characteristic equations are given by

$$\frac{da_{C_1}}{dt} = ra + w - c^w(a), \tag{5.21a}$$

$$\frac{da_{C_2}}{dt} = ra + b - c^b(a), \tag{5.21b}$$

[49]One property of a solution to the FPE (5.4) is that the mass of the initial condition for $p(a,t)$ is conserved over time (cf. BW13).

$$\frac{dp^w(a,t)}{dt} = \left[c_a^w(a) - r - s\right] p^w(a,t) + \mu p^b(a,t), \tag{5.21c}$$

$$\frac{dp^b(a,t)}{dt} = sp^w(a,t) + \left[c_a^b(a) - r - \mu\right] p^b(a,t), \tag{5.21d}$$

where the indices of a_{C_1} and a_{C_2} indicate the fact that those describe the characteristics and not the grid points of Δa. All parameters are assumed to be given and fixed. Using those parameters, optimal policy rules $c^w(a)$ and $c^b(a)$ are implied and hence assumed to be known (cf. section 5.2.2), too.

To determine a unique solution we require initial conditions. Let those conditions be defined by

$$p^z(a,0) = c_0^z(a), \; z \in \{w.b\}.$$

Having not been able to find a closed form solution in the easier case of CARA utility, we focus solely on the numerical application of the MoC according to algorithm 5.1. Once we have computed an exemplary solution in section 5.5.1, we will have a closer look on the impact of each parameter in a comparative statics analysis in section 5.5.2.

5.5.1 A generic solution

Figure 5.3 shows an exemplary solution to the problem described in eq. (5.20), where we have chosen all model parameters to show economic implications in a most clearly way. Wage w corresponds to a monthly income of 1000\$ and b corresponds to a monthly income of 100\$. Further we assume an annual interest rate of 5%, $\rho = 0.03$, and $\sigma = 1.5$. Finally, we assume initial distributions for $p^w(a,t_0) \sim N(10,2)$ and $p^b(a,t_0) \sim N(-2,5)$. The initial unemployment rate is set to $\omega_b = 0.2$. In order to determine the transition rates, we assume the average employment spell to last for five years and an unemployment spell is assumed to last on average for three years.

Compared to our first example shown in figure 5.2, we now use lower unemployment benefits and a longer unemployment spell. Those two facts increase the uncertainty in our model. Additionally, we assume that wealth is initially described according to two different distributions for employed and unemployed individuals. Choosing different means for the standard normal distribution allows easier identification of the precautionary savings effect.

Looking at figure 5.3, the results are shown in the same way as in figure 5.2. Before we start with the description, we would like to stress that the bimodal shape of the distribution in panel C as well as for later time points in panel A and B are only due to the initial choice of $p^z(a,t_0)$. Empirical wealth distributions are more likely to be unimodal as shown in the CARA case.

First, looking at $p^w(a,t)$ the main peak from the initial distribution moves upwards, i.e. from $10,000\$$ to around $30,000\$$, in the period of eight years. Responsible for this movement are the employed individuals, especially those who never became unemployed. As discussed in chapter 4 of this thesis, individuals increase their wealth only during employment. At the same time $p^w(a,t)$ gets "blown up" at the lower wealth levels. This increase in mass can be explained by individuals moving from unemployment into employment. By construction, i.e. with our chosen initial distribution $p^b(a,t_0)$, unemployed individuals have in the mean a lower wealth and as they reduce their wealth according to

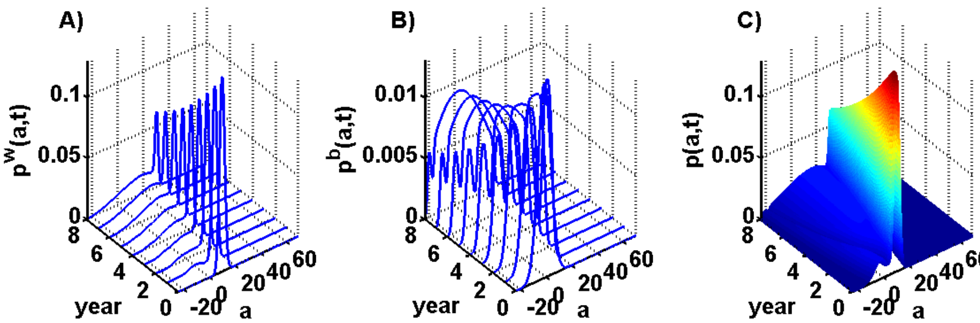

Figure 5.3: Numerical solution computed for a time span of eight years. Panel A and B show sub-densities $p^w(a,t), p^b(a,t)$ for beginning of each year only. Panel C shows unconditional density $p(a,t)$ as computed using a time grid of 0.4 months, i.e. 12 days.

the optimal consumption path they reduce their assets even more. So once they become employed, they "enter" into $p^w(a,t)$ at the lower end of the distribution.

The influence of our model setup is also visible for $p^b(a,t)$. We observe a downward shift of the initial peak as well as $p^b(a,t)$ getting broader. The first effect is due to long-term unemployed individuals who dis-save, i.e. reduce their wealth. The widening of the distribution can be explained by employed and richer individuals becoming unemployed over time and hence are represented in $p^b(a,t)$, too. Also in this case the movement of the initial peak is mainly driven by individuals staying in their initial employment status, i.e. the smaller peak moves downwards in terms of wealth from 5,000\$ to −20,000\$ due to long-term unemployed.

As $p(a,t)$ is the sum of $p^w(a,t)$ and $p^b(a,t)$ the previous results hold in panel C, too. For figure 5.3 we have chosen the normal distribution as initial condition to show the movements of the two components more clearly. A more realistic shaped wealth distribution would result once we use initial conditions with a higher degree of skewness and an upper tail.

Another implication of our model is the evolution of the unemployment rate. Looking at the integral value of $p^b(a,t)$ allows us to observe the unemployment rate directly. Figure 5.4 shows the integral value of $p^w(a,t), p^b(a,t)$ and $p(a,t)$ for the same specifications used as in figure 5.3.

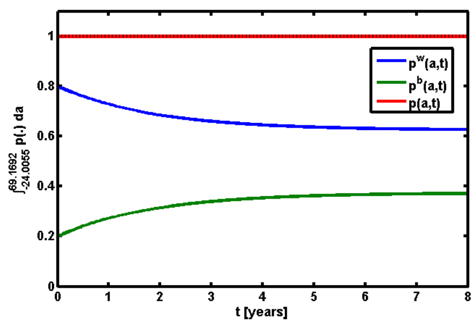

Figure 5.4: Integral mass of sub-densities $p^w(a,t), p^b(a,t)$ and unconditional density $p(a,t)$ as computed using the numerical solution.

Due to our model setup and our chosen parameters, we know the theoretic value of the long-time unemployment rate, which is equal to $\frac{s}{\mu+s} = 37.5\%$. This theoretic value is approached by the green line, i.e. $\int p^b(a,t)da$, in figure 5.4. Looking at the red line, we can observe the mass conservation property of the solution $p(a,t)$ of the FPE. This is one important aspect for judging the performance of the numerical solution in terms of accuracy. Any change in the integral value away from the initial value would indicate a truncation error, as we know that any solution to eq. (5.20) is mass conserving (cf. BW13).

5.5.2 Comparative statics

At first we will analyse the influence of the time preference rate ρ on the wealth distribution. So far we used $\rho = 0.03$ to show the general behaviour of the wealth distribution. Using two additional values for ρ, we can show that the general economic interpretation of the time preference rate is replicated by our wealth distribution. In figure 5.5, we fix all parameters to the values used in the previous section and vary ρ. To be more precise we use $\rho \in \{0.025, 0.03, 0.035\}$. Hence we include a lower and a higher value of ρ as well as the value used previously. The first main difference of the solutions shown in figure 5.5

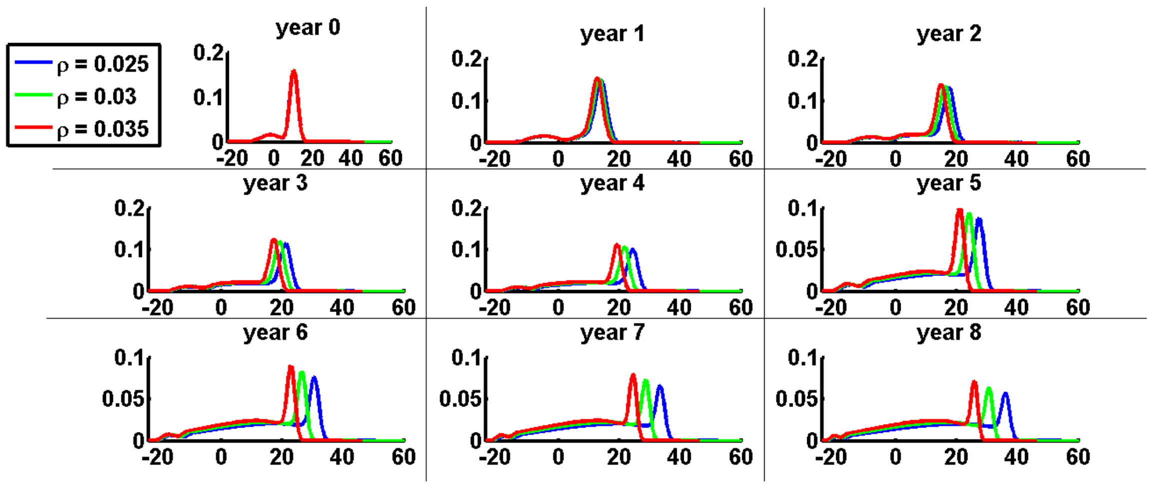

Figure 5.5: Evolution of $p(a,t)$ for three different values of ρ (shown in parts only). The green line is equivalent to results shown in figure 5.3, C.

is the change in the wealth range the model is able to reproduce, which is

$$a \in [-24.033,\ 121.006] \text{ for } \rho = 0.025,$$

$$a \in [-24.006,\ 69.192] \text{ for } \rho = 0.03,$$

$$a \in [-24.295,\ 46.282] \text{ for } \rho = 0.035.$$

As the instantaneous interest rate is equal to 0.0163, we see that the wealth range is larger the closer r and ρ are. Responsible for this change is the optimal consumption and the implied endogenous wealth range[50]. Comparing the overall shape of the different

[50] For a detailed discussion see chapter 4 of this thesis.

solutions, shows that different values of ρ do not change the general behaviour described in the previous section. The mode originating from $p^w(a, t_0)$ at $10,000\$$ moves to the right and the second "peak" originating from $p^b(a, t_0)$ at $-2000\$$ moves to the left. In between those main peaks mass increases by job-to-unemployment and unemployment-to-job transitions.

We can use several characteristics to compare different distributions. Candidates are the moments of the probability density function as well as for example the Gini coefficient or the Lorenz curve; two concepts to measure inequality of a distribution. The following figure shows the evolution of mode, median, mean, and the standard deviation over the eight year period for the three different values of ρ. Figure 5.6 shows that a higher value

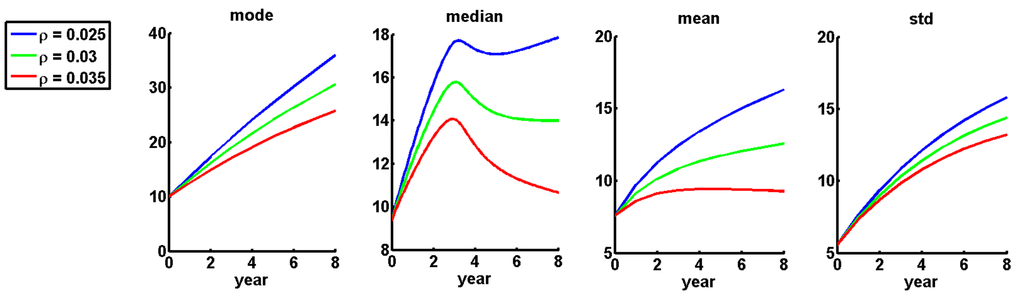

Figure 5.6: Evolution of statistical quantities densities as shown in figure 5.5

of ρ yields a lower mean wealth over time. This is in accordance to the economic intuition of the time preference rate: A higher value of ρ is characteristic for an individual focusing on his/her present situation, compared to someone with a low time preference rate, who is concerned more about his/her future. Once you focus on the present you save less, as savings are an investment for future consumption. Hence, mean wealth should be lower for higher values of ρ and this is exactly what we find in figure 5.6. Also the median, mean, and the standard deviation is lower for higher values of ρ. This can be justified with the same argument. An interesting behaviour can be seen for the median, which switches from an increase to a decline after three years. For a low time preference rate the median recovers, but for higher values of ρ the decline is permanent.

Looking at the remaining six model parameters w, b, μ, s, r, and σ and their influence, we will only show the evolution of the mean. Looking at more than one characteristic gives a better overall understanding, but in favour of a condensed presentation we abstain from presenting more figures. The general influence and interpretation of the different parameters can be given regardless.

Again we fix the initial and boundary conditions, as well as all parameters except the one we are currently analysing to their previously specified levels and execute a comparative statics analysis.

Figure 5.7 shows that all economic intuitions of the model parameters are still valid. Panel A, shows a varying yearly interest rate between 3.47% and 6.55%. A higher interest rate yields that savings are more beneficial and hence mean wealth growths faster and reaches higher levels. For the two cases of lowest interest rate it seems that the interest rate is too low for an incentive to keep up with the precautionary savings.

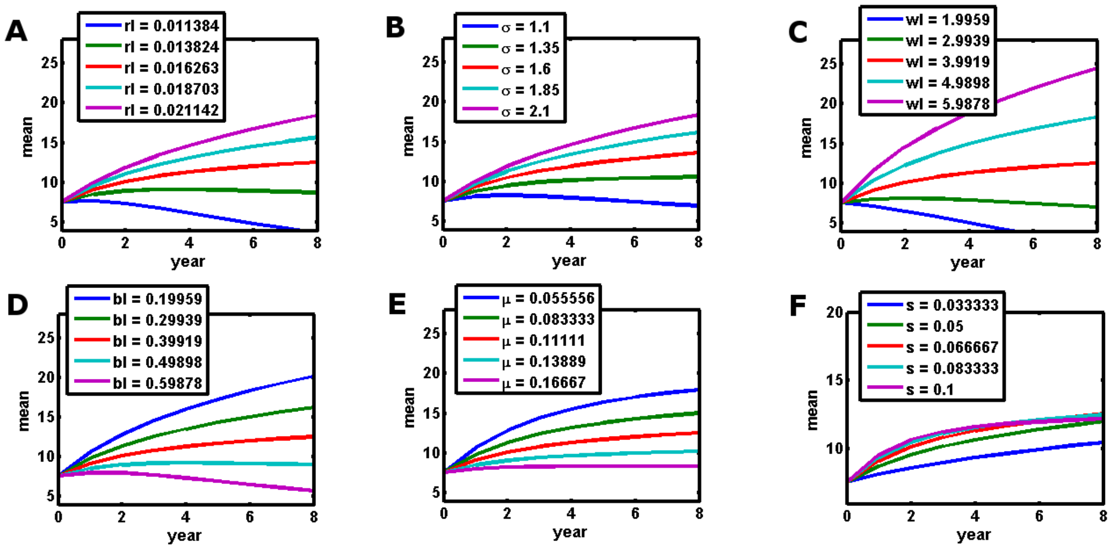

Figure 5.7: Comparative statics using the mean of $p(a,t)$. We vary parameters w, b, μ, s, r and σ in turns.(inlets of each panel gives the varying variable and the values used). All other parameters remain fixed and are chosen as described in the beginning of section 5.5.1. The red line represents the same set of parameter values in each panel.

Looking back at CRRA utility, we know that a higher value for σ is equal to a lower utility of consumption. Hence, as shown in panel B, a higher risk aversion parameter leads to less consumption and higher savings, i.e. a higher mean wealth over time.

Analysing the effects of wage income in panel C is also as expected. A higher wage increases both the speed and amount of savings leading to a higher mean wealth. For low wage rates, it seems not possible to increase individuals savings in the mean. To be more precise, mean wealth does in fact decrease over time if the wage rate is too low. Looking at the ratio between wI and bI used in panel C, the ratio wI/bI varies between 5 (for $wI = 1.9959$) and 15 ($wI = 5.9878$)[51]. Hence for a low ratio, mean wealth is lower than in the case of a higher ratio. This is also an intuitive result, taking into account the length of the (un)employment spells. Individuals can only save when employed and in case they earn more they are likely to save more.

Looking at panel D and the effects of different unemployment benefit payments can be seen. A lower value for unemployment benefits leads to higher mean wealth. This is at first sight counter-intuitive behaviour, but it can be explained using the ratio wI/bI. This ration varies between 20 (for $bI = 0.19959$) and 6.667 ($bI = 0.59878$) and hence the behaviour is similar to the one in the previous case of varying w. A high ratio wI/bI leads to higher mean wealth.

Panel E and F analyse the influence of the arrival rates. We vary the length of the unemployment spell between six years ($\mu = 0.05556$) and two years ($\mu = 0.1667$). The employment spell is constant and equal to five years. Here a shorter unemployment spell leads to a lower mean wealth. In terms of the ratio μ/s we use a range of 0.833 ($\mu = 0.0556$) and 2.5 ($\mu = 0.167$), where a lower ratio leads to a higher mean wealth. Again it seems that the ratio is more important than the arrival rates itself. This fact is validated looking at the influence of s. Here we vary the length of the employment

[51] As a reminder, we used a fixed value of $bI = 0.399187$.

spell between ten years ($s = 0.033$) and three years and four months ($s = 0.1$). As a reference, the unemployment spell is fixed to be equal to three years. In general, we see that for a longer employment spell duration mean wealth is lower than for shorter employment duration spells. This could be explained by a neglect of precautionary savings due to a higher degree of security. In case of short employment spells, i.e. a high risk of unemployment and high insecurity, individuals choose to save more and hence mean wealth increases. In terms of ratio μ/s, which varies between 3.33 ($s = 0.033$) and 1.11 ($s = 0.1$) a lower ratio leads in general to a higher mean wealth - the same result as seen in panel E[52].

In summary, comparative statics have shown that the ratios w/b and μ/s are of importance if we want to explain influences. Nevertheless, the absolute value of each variable on its own does influence the results as well.

5.6 Conclusion

The objective of this paper was to give possible solution techniques for a coupled two-dimensional system of partial differential equations arising in the analysis of distributional properties via FPEs. Focusing on an stochastic labour market model with two different specifications of individual instantaneous utility function, we are able to show that the MoC can be used to obtain a solution in both cases. In general, the MoC can be used to obtain an analytical as well as an numerical solution. In terms of a numerical solution, we compare the MoC to two finite-difference methods that can be used to solve the problem in question - the Lax-Friedrich method and the Crank-Nicolson method. Due to the additional information that can be gained from the MoC as well as the theoretical possibility of obtaining a closed form solution with the help of the MoC, we favour the MoC. Finally, we apply the MoC for both specifications of constant absolute risk aversion and constant relative risk aversion.

In the simpler case of CARA utility, we show that the MoC can be used to express the probability density function of wealth as a system of integral equation. Unfortunately, this system cannot be solved analytically. Using a numerical solution, we can present an exemplary solution of the FPEs.

For the economic more relevant and more complex version of the model with CRRA utility, we demonstrate how wealth evolves over time using again a numerical solution based on the MoC system. Especially due to the characteristics, we can explain how an initial wealth distribution changes its shape over time. Here the influence of long-time unemployed at the lower end of the distribution and the influence of employed individuals at the upper end are clearly visible. Other results obtained in a comparative statics analysis do also show economic conclusive behaviour of the numerical solution.

In summary, using the MoC in combination with the use of FPEs offers a strong tool to answer questions concerning distributional dynamics. Looking at distributional properties in general is a pervasive motive in economic research, making both of those techniques a favourable tool for future research.

[52] $s = 0.1$ and $s = 0.083$ seems to be a special case as the mean wealth is lower or equal after eight years than with lower s values. This can be explained with the close proximity of $\mu = 0.11$ and s in those cases.

5.7 Appendix of Chapter 5

5.7.1 A simple test case for the generic algorithm

From a intuitive point of view our generic algorithm is the one-to-one mapping of the mathematical idea behind the MoC to a numerical solution. Nevertheless, we want to test algorithm 5.1 with a fairly simple example, where we know all key ingredients. Therefore let $u(x,t) : \mathbb{R}^2 \times \mathbb{R}_0^+ \to \mathbb{R}$. We want to solve the following initial value problem:

$$u_t + \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} u_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad u(x,0) = \begin{pmatrix} 2x \\ 3x \end{pmatrix}.$$

It is easy to verify that

$$u(x,t) = \begin{pmatrix} 2x - 8t \\ 3x - 27t \end{pmatrix}$$

is the unique solution to that problem. Basically, we deal with two independent PDEs as they are not coupled through the right hand side.

Transforming those PDEs into ODEs with the help of the MoC reads

$$\dot{a}_{C_1} = 4, \qquad \dot{a}_{C_2} = 9, \tag{5.22a}$$

$$\dot{u_1}\left(x_1\left(t\right),t\right) = 0, \qquad \dot{u_2}\left(x_2\left(t\right),t\right) = 0. \tag{5.22b}$$

Those equations are all easy to solve. The characteristics are given by $a_{C_1}(t;x_i) = x_i + 4t$ and $a_{C_2}(t;x_i) = x_i + 9t$. Eqs. (5.22b) can also be solved explicitly, i.e. $u_i(.) = C_i,\ i = 1,2$ and C_i constant. Putting those results together, the MoC tells us that the solution of the PDEs is constant along a straight line. Once we know an initial value, this value is propagated along the characteristic, i.e. along a straight line.

According to algorithm 5.1, we have to compute the solution of the ODEs (here a constant value given by the previous step) and than we "move" with this solutions along the characteristics. Afterwards, we compute the values of the solutions on the original grid due to interpolation (and extrapolation). In our example the exact solutions are two planes in $\mathbb{R}^3$. The solution computed with "eEuler" produces already good results for small values of h_t due to the fact that the solution of the ODE is constant. This is a result with respect to the relative error, i.e. the quotient of the absolute value of the difference between the approximation and the exact value and the absolute value of the exact solution, which is shown in figure 5.8. Therein, the error values around the dash-dotted line, i.e. the roots of the exact solution, can be explained by the nature of the relative error. Whenever the exact solution equals 0, i.e. in our case at $x = 4t$ for $u_1(.)$ and at $x = 9t$ for $u_2(.)$, the relative error is not defined as we would need to divide by zero. Also in the proximity of this line, the relative error is likely to "explode" as soon as the numerical approximation differs slightly from 0 as we divide by a value close to zero.

Besides that problem, the numerical solution is approximate up to 10^{-10} % and 10^{-9} % respectively. Hence we can claim that the exact solution and the approximation are identical and our algorithm is correct.

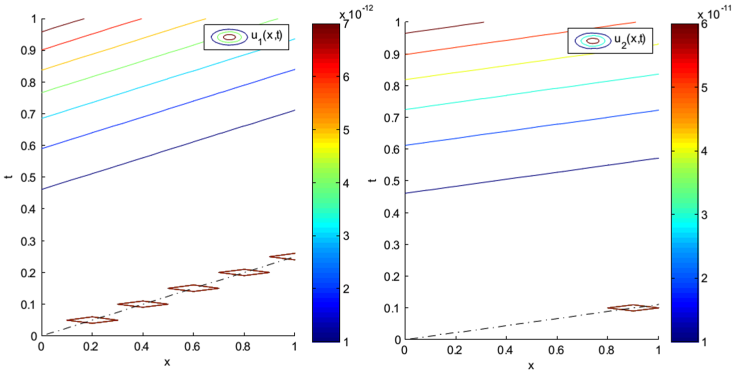

Figure 5.8: Contourplot of the relative error. The dash-dotted line represents the loci where the exact solution is equal to 0*. Along this line the relative error is likely to explode if the numerical solution is not equal to 0.*

5.7.2 Derivation of the Crank-Nicolson method

This section gives the derivation of the formulas needed for the Crank-Nicolson method. The partial derivatives are approximated using

$$\begin{aligned}
\frac{\partial}{\partial a} p^z_{i,j+1} &= \frac{1}{2}\left[\frac{p^z_{i+1,j+1} - p^z_{i-1,j+1}}{2h_a} + \frac{p^z_{i+1,j} - p^z_{i-1,j}}{2h_a}\right] \\
&= \frac{1}{4h_a}\left[p^z_{i+1,j+1} - p^z_{i-1,j+1} + p^z_{i+1,j} - p^z_{i-1,j}\right], \\
\frac{\partial}{\partial t} p^z_{i,j+1} &= \frac{p^z_{i,j+1} - p^z_{i,j}}{h_t}, \\
p^z_{i,j+1} &= \frac{1}{2}\left(p^z_{i,j+1} + p^z_{i,j}\right), \ z \in \{w, b\}.
\end{aligned}$$

Again assuming that we know all values at time t_j the PDE approximating the value at (a_i, t_{j+1}), $i \in [1, ..., m]$ reads

$$\begin{aligned}
\frac{p_{i,j+1} - p_{i,j}}{h_t} + \frac{1}{4h_a}\begin{pmatrix} B_{11} & 0 \\ 0 & B_{22}\end{pmatrix}&\left[p_{i+1,j+1} - p_{i-1,j+1} + p_{i+1,j} - p_{i-1,j}\right] \\
&= \frac{1}{2}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{pmatrix}\left(p_{i,j+1} + p_{i,j}\right).
\end{aligned}$$

Using $s := \frac{h_t}{h_a}$ this can be simplified to

$$\begin{aligned}
4\left(p_{i,j+1} - p_{i,j}\right) + s\begin{pmatrix} B_{11} & 0 \\ 0 & B_{22}\end{pmatrix}&\left[p_{i+1,j+1} - p_{i-1,j+1} + p_{i+1,j} - p_{i-1,j}\right] \\
&= 2h_t\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22}\end{pmatrix}\left(p_{i,j+1} + p_{i,j}\right).
\end{aligned}$$

Looking at the first row of this matrix equation and rearranging values according to their time evaluation yields:

$$\begin{aligned}
4p^w_{i,j+1} + &sB_{11}\left(p^w_{i+1,j+1} - p^w_{i-1,j+1}\right) \\
&= 4p^w_{i,j} - sB_{11}\left(p^w_{i+1,j} - p^w_{i-1,j}\right) + 2h_t\left[C_{11}\left(p^w_{i,j+1} + p^w_{i,j}\right) + C_{12}\left(p^b_{i,j+1} + p^b_{i,j}\right)\right].
\end{aligned}$$

Using simple algebra this is equivalent to

$$\begin{aligned}-sB_{11}p^w_{i-1,j+1}+&\left(4-2h_tC_{11}\right)p^w_{i,j+1}+sB_{11}p^w_{i+1,j+1}-2h_tC_{12}p^b_{i,j+1}\\ &=sB_{11}p^w_{i-1,j}+\left(4+2h_tC_{11}\right)p^w_{i,j}-sB_{11}p^w_{i+1,j}+2h_tC_{12}p^b_{i,j}.\end{aligned}$$

Doing the same for the second row yields:

$$\begin{aligned}-sB_{22}p^b_{i-1,j+1}+&\left(4-2h_tC_{22}\right)p^b_{i,j+1}+sB_{22}p^b_{i+1,j+1}-2h_tC_{21}p^w_{i,j+1}\\ &=sB_{22}p^b_{i-1,j}+\left(4+2h_tC_{22}\right)p^b_{i,j}-sB_{22}p^b_{i+1,j}+2h_tC_{21}p^w_{i,j}.\end{aligned}$$

The objective is to compute all values of p at time t_{j+1} given our knowledge of the boundary condition (i.e. $p_{1,j+1}$ and $p_{m,j+1}$) as well as all previously computed values $p_{i,j},\ i=1,...,m$.

Introducing a vector containing the unknowns we want to determine for t_{j+1}, i.e. $\hat{p}_{i,j+1} := \left[p^w_{2,j+1},...,p^w_{m-1,j+1},p^b_{2,j+1},...,p^b_{m-1,j+1}\right]$, we end up with the following system that needs to be solved:

$$L\cdot\hat{p}_{i,j+1}=z,\quad L\in\mathbb{R}^{2(m-2)\times 2(m-2)}$$

where

$$L\equiv\begin{pmatrix}
4-2h_tC_{11} & sB_{11} & & & -2h_tC_{12} & & & \\
-sB_{11} & \ddots & \ddots & & & \ddots & & \\
 & \ddots & \ddots & sB_{11} & & & \ddots & \\
 & & -sB_{11} & 4-2h_tC_{11} & & & & -2h_tC_{12} \\
-2h_tC_{21} & & & & 4-2h_tC_{22} & sB_{22} & & \\
 & \ddots & & & -sB_{22} & \ddots & \ddots & \\
 & & \ddots & & & \ddots & \ddots & sB_{22} \\
 & & & -2h_tC_{21} & & & -sB_{22} & 4-2h_tC_{22}
\end{pmatrix}$$

and

$$z\equiv\begin{pmatrix}
sB_{11}\left(p^w_{1,j}-p^w_{3,j}+p^w_{1,j+1}\right)+\left(4+2h_tC_{11}\right)p^w_{2,j}+2h_tC_{12}p^b_{2,j}\\
sB_{11}\left(p^w_{2,j}-p^w_{4,j}\right)+\left(4+2h_tC_{11}\right)p^w_{3,j}+2h_tC_{12}p^b_{3,j}\\
\vdots\\
sB_{11}\left(p^w_{m-3,j}-p^w_{m-1,j}\right)+\left(4+2h_tC_{11}\right)p^w_{m-2,j}+2h_tC_{12}p^b_{m-2,j}\\
sB_{11}\left(p^w_{m-2,j}-p^w_{m,j}-p^w_{m,j+1}\right)+\left(4+2h_tC_{11}\right)p^w_{m-1,j}+2h_tC_{12}p^b_{m-1,j}\\
sB_{22}p^b_{1,j}+\left(4+2h_tC_{22}\right)p^b_{2,j}-sB_{22}p^b_{3,j}+2h_tC_{21}p^w_{2,j}+sB_{22}p^b_{1,j+1}\\
sB_{22}p^b_{2,j}+\left(4+2h_tC_{22}\right)p^b_{3,j}-sB_{22}p^b_{4,j}+2h_tC_{21}p^w_{3,j}\\
\vdots\\
sB_{22}p^b_{m-3,j}+\left(4+2h_tC_{22}\right)p^b_{m-2,j}-sB_{22}p^b_{m-1,j}+2h_tC_{21}p^w_{m-2,j}\\
sB_{22}p^b_{m-2,j}+\left(4+2h_tC_{22}\right)p^b_{m-1,j}-sB_{22}p^b_{m,j}+2h_tC_{21}p^w_{m-1,j}-sB_{22}p^b_{m,j+1}
\end{pmatrix}.$$

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