This is chapter 4 of the dissertation "Lessons Learned From Germany’s 2001-2006 Labor Market Reforms" by Irene Schumm, defended in December 2009 at the University of Würzburg. Ch. 4.4.2 describes in detail how the numerical solution of the Volterra equations in Launov and Wälde was obtained.

Chapter 4

Semi-Markov processes in labor market theory

4.1 Introduction and underlying setup

Semi-Markov processes are, like all stochastic processes, models of systems or behavior. As extensions of Markov processes and renewal processes, Semi-Markov processes are widely applied and hence, an important methodology for modeling. Semi-Markov processes are used in computer science and engineering, e.g. in queuing theory and server models, see Cohen (1982). In finance, for example, credit rating and reliability models are based upon Semi-Markov theory like in D’Amico et al. (2006). Other applications in business administration are operations research like in Sobel and Heyman (2003), as well as manpower models as described in Mehlman (1979). Moreover, Semi-Markov models are employed in sociology or socioeconomics, see Mills (2004) for a model of the marriage market. In biology and medicine, Semi-Markov processes are used for prognosis and the evolution of diseases, see Beck and Pauker (1983) or Foucher et al. (2005). For demographic questions, models of disability or fertility, Semi-Markov processes are employed, too, see Hoem (1972).

Consequently, Semi-Markov processes are interdisciplinary important and, of course, also economics has discovered the usefulness for modeling issues. Already Markov processes, which can be seen as a special case of Semi-Markov processes, are widely used to describe the different states of an economy or an individual. Depending on the currently occupied state only, there are different transition rates to other states. Possible applications of Markov chains in economics are standard matching models of the labor market as described in Pissarides (2000) or money demand models like in Kiyotaki and Wright (1993). In this chapter, we will focus on the former ones as the methods presented build the background for the numerical solution of our labor market model in chapter 3. Typically, the possible states of an individual in the labor market are *unemployment* or *employment* and the transitions between these states are described by Markov processes. For simplification, most of the models in literature take the transition rates between the labor market states to be constant, see the standard matching setup in Pissarides (2000), Pissarides (1985), Mortensen and Pissarides (1994), for example, or Rogerson et al. (2005) for an overview. This simplification may be appropriate for many questions if incentive effects of labor market institutions can be neglected. For other applications, however, this assumption needs generalization. When the behavior of individuals and the incentive effects of unemployment insurance systems are to be analyzed, for example, stationary job arrival rates over the unemployment spell are no longer realistic, see Mortensen (1977) amongst others. In fact, it is plausible that the arrival rate of jobs exhibits true duration dependence. Reasons for this can be found in search effort reactions due to non-stationary benefits or stigmas attached to or perceived by long-term unemployed. Empirical evidence with respect to non-stationary hazard rates can be found in Heckman and Borjas (1980), Meyer (1990), or van den Berg and van Ours (1994), for instance.[1] However, models considering duration-dependent hazard rates are typically restricted to analyze microeconomic behavior only and thus, the Semi-Markov structure is negligible as the first order condition for optimal behavior is unaffected. Therefore in chapter 3, a full equilibrium labor market model is built up with non-stationary exit rates out of unemployment and the parameters of the model are estimated structurally.

Allowing for duration-dependent transition rates has methodological consequences regarding the state distribution of individuals. Analytical solutions for transition probabilities and distributions are no longer feasible for such models having non-analytic and non-stationary transition rates and numerical solution methods are required. Thus, the purpose of this chapter is twofold. First, an accurate, but intuitive definition and classification of Semi-Markov processes among the family of stochastic processes will be given, emphasizing the application to labor market models. Second, it provides a recipe of how to solve for the transition probabilities of Semi-Markov processes, as well as the description of the limiting behavior.

In a first step, this chapter presents the Semi-Markov theory. The properties and transition probabilities, as well as the limiting behavior are discussed on the basis of Pyke (1961a) and Pyke (1961b), Kulkarni (1995), and Ross (1996). While the transition probabilities of continuous-time Markov chains are computed using the Chapman-Kolmogorov equations, which can be solved analytically, for Semi-Markov processes, the correspond-

[1] For a discussion of non-stationary hazard rates and possible sources, see subsection 3.2.2.

ing probabilities are based on the renewal argument and convolution theory. An analytical solution is very difficult in this case and impossible for the setup of chapter 3, so the determination of the transition probabilities and of the limiting probabilities is about numerical solution methods and it makes sense to deal with a specific example. Considering the economic model of chapter 3, there exist two groups in the labor market like in the standard model: the unemployed and the employed workers. This makes things as simple as possible, but clearly shows the solution approach at the same time. The job of an employed worker is destroyed at an exogenous separation rate λ and so the waiting time until job destruction is exponentially distributed with parameter λ. An unemployed job seeker with unemployment spell s gets new offers at rate $\mu\left(\phi\left(s\right)\theta,\eta\left(s\right)\right)$, where $\phi\left(s\right)$ is the job search effort of the unemployed with spell s, θ is the labor market tightness, and $\eta\left(s\right)$ is an exogenous spell effect.[2] Having an unemployment insurance system with non-stationary benefits, it makes sense to assume that an unemployed individual adjusts his search effort over the spell. With increasing unemployment duration, for example, the lower benefits of long-term unemployed get closer. Thus, it is plausible that effort increases before long-term unemployment is realized. Assuming that the job arrival rate $\mu\left(\phi\left(s\right)\theta,\eta\left(s\right)\right)$ increases with search effort, this partial effect would lead to an increasing job arrival rate. The duration-dependent spell effect $\eta\left(s\right)$ catches remaining duration-dependent factors, which may affect the job arrival rate. This partial effect is discussed in chapter 3 in detail, where it leads to a decreasing job arrival rate for long-term unemployed.

In this chapter, however, we focus on the pure duration dependence and not on its sources. Therefore, we neglect all other arguments but s and reduce the notation to $\mu\left(s\right)$ for simplification.

In chapter 3, the steady state behavior of the model economy is analyzed. Using the optimal search effort of an unemployed over the unemployment spell, we derive the densities for the duration in both states. With these densities, the parameters of the structural arrival rate are estimated with micro data from the GSOEP. Based on the parameter estimates, the job arrival rates can be computed as well as transition probabilities and hence, the state distribution for an economy of representative agents can be determined applying the methods derived in this chapter. The knowledge of the state distribution makes it possible to evaluate the Hartz IV reforms in terms of unemployment and welfare effects by models like the one in chapter 3.

The starting point for the calculation of the transition probabilities are interdependent Volterra integral equations of the first and the second kind, which can be derived applying

[2]Compare subsection 3.3.1 for details on the modeling of the job arrival rate.

the Semi-Markov theory. The key issue is to solve the integrals, which contain unknowns and cannot be solved analytically. To this end, the problem is transformed into a discrete one and numerical solution methods are discussed. The different methods have different advantages and drawbacks. As a rule, the more precise a method is, the longer the computation takes, leading to a time-preciseness trade-off. The different numerical results for the transition probabilities are therefore collected and discussed. First, the special case of constant arrival rates is considered. The Semi-Markov process is a continuous-time Markov chain then, for which the transition probabilities are known. Hence, the numerical solutions can directly be compared to the analytical solution. Permitting non-stationary arrival rates, with the setup taken from chapter 3, a comparison to an analytical solution is no longer possible. Hence, the numerical methods can only be studied independently and with respect to the limiting behavior. As expected, the more complicated method provides the more exact results for the transition probabilities. Since this already applies to smaller step numbers, the computational effort of the more complex method can be outweighed by using less steps. This also applies to the limiting distribution.

The outline of this chapter is as follows. Section 4.2 describes the basics of Semi-Markov processes. From section 4.3 on, we apply the Semi-Markov theory to our labor market model presented in chapter 3, in order to illustrate solution procedures for transition probabilities of Semi-Markov processes. In section 4.4, numerical solution procedures are described. Section 4.5 presents and compares the outcomes of the different numerical methods and section 4.6, finally, concludes with the findings of this chapter.

4.2 Semi-Markov processes - the basics

This section deals with the basics of (Semi-)Markov processes. First of all, like Markov processes, a Semi-Markov process is a stochastic process. A stochastic process collects realizations of one or more random variables over time and the theory of stochastic processes tries to find models which describe such probabilistic systems. One can distinguish between discrete-time processes and continuous-time processes. While the system is observed at discrete points in time only in the first case, there is continuous observation given for the latter. Throughout this chapter, we focus on the continuous-time versions. The starting point of this section is a brief introduction to Markov processes since many well-known concepts also hold for Semi-Markov processes. After that, the definition of Semi-Markov processes will be given and their properties will be outlined. The section concludes with a derivation of conditional transition probabilities of Semi-Markov processes.

4.2.1 Continuous-time Markov chains

Markov chains are stochastic processes and have the property of being memoryless. This means that a continuous-time Markov chain (CTMC) is a sequence of realized states and the transition probability to another state depends on the current state only and not on the history of states. Therefore, for the continuous-time Markov chain the following Markov property holds:

$$P\{X(t+s)=j|X(t)=i, X(u):0\le u<t\} = P\{X(t+s)=j|X(t)=i\},$$

where $X(t)$ denotes the state of the system at time t and $X(u): 0 \le u < t$ denotes all states $X(u)$ in the history from 0 up to t, compare Kulkarni (1995). In other words, this property means that the probability of being in state j at $t+s$, given that the system was in state i at t and the complete history of states, is equal to the probability without the information on the complete history.

The duration period of a CTMC in state i is exponentially distributed with parameter λ_i, so the probability of leaving a state i towards another *arbitrary* state in a spell of s or less is given by

$$F(s) = P\{S \le s\} = \begin{cases} 0 & \text{if } x \le 0 \\ 1 - e^{-\lambda_i x} & \text{if } x > 0. \end{cases}$$

The state duration period of leaving state i towards a *specific* state j is exponentially distributed with parameter $\lambda_{ij} \ge 0$. By definition, it holds that $\sum_{j\neq i} \lambda_{ij} = \lambda_i$. The parameters λ_i and λ_{ij} are also called transition or hazard rates, which becomes clear when considering the definition of the hazard. The hazard rate is the probability of instantaneously leaving state i at t, given that state i has been occupied till t, see Lancaster (1990). Therefore, the hazard rate for leaving state i to any state is the probability density function of the duration $f(t)$ divided by the survival function in this state i, $1-F(t)$:

$$\begin{aligned} h(t) &= \frac{f(t)}{1-F(t)} \\ &= \frac{\lambda_i e^{-\lambda_i t}}{e^{-\lambda_i t}} = \lambda_i. \end{aligned}$$

Equivalently, the hazard rate λ_{ij} for leaving state i and going to state j can be determined. The states of a Markov process and the corresponding transition rates can be visualized in rate diagrams. Figure 4.1 shows the rate diagram for a two-state Markov process. Let the states be state '0' and state '1' and the transition rates λ_{01} and λ_{10} be given by μ and λ[3], respectively. Clearly, in a process with two states, the rate of leaving i and going to

[3]The variable λ_i with a subscript denotes general arrival rates, while the variable λ without any subscripts is often used for separation rates in job search models. This notation is kept throughout this chapter.

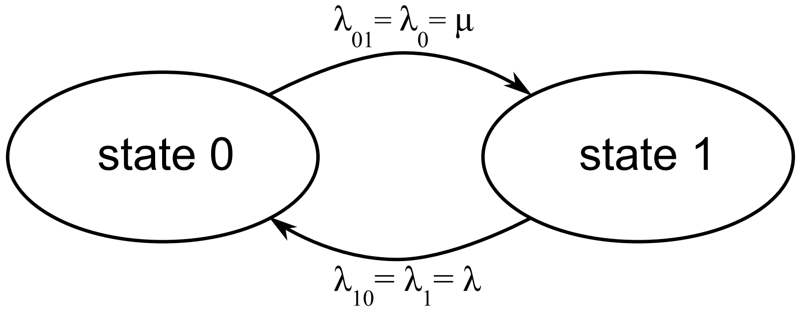

Figure 4.1: Rate diagram for a CTMC with two states (0 and 1). The states are represented by the ovals. The transition rates are given at the arrows that symbolize the transition.

j, λ_{ij}, is identical to the rate of leaving i, $\lambda_i = \sum_{j \neq i} \lambda_{ij}$. Therefore, the rates are simply given by $\lambda_{01} = \lambda_0 = \mu$ and $\lambda_{10} = \lambda_1 = \lambda$.

The transition probability matrix $P = [p_{ij}(t)]$ contains the probabilities that the system which is initially in state i will be in state j at t, $P\{X(t) = j | X(0) = i\}$. In order to compute these transition probabilities, the Chapman-Kolmogorov equations can be used, for details see Ross (1996), for instance. In contrast to discrete-time Markov chains (DTMCs), where the limiting behavior depends on specific properties of the DTMC, the limit of a CTMC transition probability matrix always exists. The limits are given by

$$\lim_{t \to \infty} p_{jj}(t) = \frac{1}{\lambda_j \eta_{jj}}$$

and

$$\lim_{t \to \infty} p_{ij}(t) = \frac{f_{ij}}{\lambda_j \eta_{jj}},$$

where f_{ij} is the probability that the spell of state i is less than infinity and a transition occurs to j, $f_{ij} = P\{T_j < \infty | X(0) = i\}$. T_j is the first time the CTMC enters state j and η_{jj} is the expected reoccurrence time of state j, given that the initial state is j, $\eta_{jj} = E[T_j | X(0) = j]$. A proof is provided in Kulkarni (1995).

The interpretation of the limit of $p_{jj}(t)$ is as follows: $1/\lambda_j$ is the expected duration in state j and once the process leaves state j, η_{jj} is the expected time until re-entering state j.

For the limiting probability of ending in j when starting in i, one needs to know how likely a transition from i to j in a period less than infinity is, which is given by $f_{ij} = P\{T_j < \infty | X(0) = i\}$. Once the system enters state j, only the limiting probability for ending in state j upon beginning in state j is needed, which we just determined as $p_{jj}(t) = 1/(\lambda_j \eta_{jj})$. The joint probability is then the product of both probabilities, therefore f_{ij} is multiplied by $1/(\lambda_j \eta_{jj})$.

The limiting probabilities are illustrated by returning to the example from figure 4.1. The rate λ_1 is given by $\lambda_1 = \lambda$ and the rate λ_0 by $\lambda_0 = \mu$. The expected reocurrence time η_{jj} is given by the sum of the expected duration in both states, $\eta_{jj} = \frac{1}{\lambda_j} + \frac{1}{\lambda_i}$. So, the expected duration in state j, $1/\lambda_j$, and the expected duration in state i, $1/\lambda_i$, after having left state j are added up. Having all this in mind, $\lim_{t\to\infty} p_{jj}(t) = \frac{1}{\lambda_j \eta_{jj}}$ becomes[4]

$$\lim_{t\to\infty} p_{00}(t) = \frac{\lambda}{\mu + \lambda} \text{ and } \lim_{t\to\infty} p_{11}(t) = \frac{\mu}{\mu + \lambda}. \tag{4.1}$$

In standard labor market models with the two states *employment* and *unemployment*, this limiting distribution is equal to the equilibrium unemployment rate and employment rate, respectively, which can be shown by using a law of large numbers.

CTMCs whose expected returning time for a state is less than infinity are called *ergodic* and they have an interesting property. Namely, the limiting distribution of the states does not depend on the initial distribution of states, $p_j = \lim_{t\to\infty} P\{X(t) = j | X(0) = i\}$. In this case, the limiting distribution can be computed by using the so-called balance equations,

$$\sum_{j\epsilon S} p_i \lambda_{ij} = \sum_{j\epsilon S} p_j \lambda_{ji},$$

combined with the condition that all probabilities must sum up to 1, $\sum_{j\epsilon S} p_j = 1$. The idea behind the balance equation is quite simple: in the limit, flows out of state i must equal flows into state i. This property also leads to the well-known expression for the equilibrium unemployment rate in standard matching models with constant arrival rates.

4.2.2 Semi-Markov processes

Also for Semi-Markov processes (SMPs) it holds that only the current state is relevant for the transition rates - and in this sense, there is still memorylessness. However, the transition rates to other states may change over the duration of a state and therefore, the inter-arrival times between subsequent states are no longer exponentially distributed. Thus, the extensions compared to CTMCs are an arbitrary duration distribution and non-stationary transition rates.

A natural way to approach SMPs is through renewal theory, where inter-arrival times between events do not need to be exponentially distributed. For this purpose, it is helpful to define a Markov renewal sequence as a sequence of a bivariate random variable first.

[4] In a system with two states, the remaining limiting probabilities are computed by $\lim_{t\to\infty} p_{ij}(t) = 1 - \lim_{t\to\infty} p_{ii}(t)$. Hence, the limiting transition probability from state 1 to state 0 is $\lim_{t\to\infty} p_{10}(t) = \frac{\lambda}{\mu+\lambda}$ and the limiting transition probability from state 0 to state 1 is $\lim_{t\to\infty} p_{01}(t) = \frac{\mu}{\mu+\lambda}$.

The two elements of this bivariate random variable are the observation time S_n of the nth transition and the corresponding nth observation Y_n, $n \geq 0, Y_n \epsilon I = \{0,1,2,..\}$. The joint probability of observing $Y_{n+1} = j$ in an inter-arrival time of $S_{n+1} - S_n \leq x$, conditioned on the observation history, satisfies the Markov property,

$$P\{Y_{n+1} = j, S_{n+1} - S_n \leq x | Y_n = i, S_n, Y_{n-1}, S_{n-1}, ..., Y_0, 0\} = P\{Y_{n+1}, S_{n+1} - S_n \leq x | Y_n = i\} \equiv G_{ij}(x). \quad (4.2)$$

Finally, a SMP is a stochastic process that records the state of the Markov renewal process at each point in time, see Pyke (1961a).
More formal, let $\{(Y_n, S_n), n \geq 0\}$ be a Markov renewal sequence. Let $N(t)$ be the state with the last completed state spell before t, $N(t) = \sup\{n \geq 0 \colon S_n \leq t\}$, and let $X(t) = Y_{N(t)}$. Then, the stochastic process $\{X(t), t \geq 0\}$ is denoted as a Semi-Markov process. The matrix $G(x) = [G_{ij}(x)]$ as defined in equation (4.2) is called the *kernel* of the SMP, compare Kulkarni (1995).

Next, we discuss some properties of SMPs, which help to classify them. A SMP is *time-homogeneous* if just the interval until the next transition matters for the probability - not when this interval started, or more specific

$$P\{Y_{n+1} = j, S_{n+1} - S_n \leq x | Y_n = i\} = P\{Y_1 = j, S_1 \leq x | Y_0 = i\}.$$

A SMP is called *regular* if there is only a finite number of transitions possible in a finite time period. The SMP is *irreducible* if each state can be reached from any other state; the states are said to communicate with each other in this case. A state j is called *recurrent* if the process returns to this state j in a spell less than infinity and it is called *transient* otherwise (if it never returns). A state is denoted as *positive recurrent* if it is recurrent and the expected returning time to state i, given the process started in i, is less than infinity. For a SMP, a recurrent state i is called *aperiodic* if it is possible to visit this state anytime. *Periodicity* with period d is given if a state i can only be visited at positive multiple integers of d, $d > 1$, see Ross (1996). Therefore, aperiodicity actually means $d = 1$. The initial distribution vector of states $a = [a_i]$ reports the probability that the state of the system is i at the beginning, $a_i = P\{X(0) = i\}$. Finally, a regular SMP is fully specified by the initial distribution of states a and the kernel $G(x) = [G_{ij}(x)]$.

Example. In standard labor market models with two states, all states in the SMP communicate. Furthermore, the SMP is regular, positive recurrent, irreducible, and finally, aperiodic. It is intuitive why: the state *unemployment* is accessible from the state *employment* and vice versa. Hence, the states communicate and the SMP is irreducible. The SMP is regular because the probability of very short durations is less than one. This means that finding a job or loosing it normally needs some time. It is positive recurrent because the expected 'revisiting' duration for an unemployed or an employed is less than infinity. The SMP is aperiodic because obviously $d = 1$ in this two-state process.

Deriving the conditional distribution of the states in a SMP $\{X(t), t \geq 0\}$ at a fixed $t \geq 0$ requires something like the Chapman-Kolmogorov equations, but for SMPs. In doing so, the renewal argument is used to develop integral equations, which is postponed to the next subsection. The numeric methods described in the remainder of this chapter then deal with the computation of these integral equations.

For positive recurrent, irreducible, and aperiodic SMPs, the limiting probability of being in state j when starting in state i is independent of i,

$$p_j = \lim_{t\to\infty} P\{X(t) = j | X(0) = i\} = \frac{\pi_j \eta_j}{\sum_{k=0}^{\infty} \pi_k \mu_k}, \tag{4.3}$$

where π is a solution to $\pi = \pi G(\infty)$ and η_k is the expected duration in state k, $k = 0, 1, 2, ...$, see Kulkarni (1995); also a proof is provided there.

For a labor market model with the two states 1 (employment) and 0 (unemployment), the kernel is given by $G_{10}(\infty) = 1$ and $G_{01}(\infty) = 1$, hence $\pi = (1, 1)$ satisfies the equation $\pi = \pi G(\infty)$. Therefore, equation (4.3) becomes $p_0 = \frac{\eta_0}{\eta_0 + \eta_1}$. The limiting probability of being unemployed is given by the expected duration of the state unemployment divided by the sum of the expected duration in the two states unemployment and employment. According to Cox (1962), this holds for any distribution.
Consequently, the limiting distribution in a two-state labor market model, with duration-dependent transition rates $\mu(.)$ and $\lambda(.)$, becomes[5]

$$p_0 = \frac{\int_0^\infty \exp\left\{-\int_0^x \mu(v)\,dv\right\} dx}{\int_0^\infty \exp\left\{-\int_0^x \lambda(v)\,dv\right\} dx + \int_0^\infty \exp\left\{-\int_0^x \mu(v)\,dv\right\} dx}, \tag{4.4}$$
$$p_1 = 1 - p_0.$$

Equipped with this intuitive, but also formal classification of Semi-Markov processes, the next subsection describes the derivation of the transition probabilities with the integral equations mentioned above.

4.2.3 Transition probabilities of Semi-Markov processes

Now we turn to the transition probabilities of SMPs. This subsection states the general notation and the mathematical basics used throughout this chapter when computing the conditional transition probabilities of a SMP. Pyke (1961a) and Pyke (1961b) are the

[5] See appendix chapter B.1 for a derivation.

seminal articles mentioned in nearly every work about Semi-Markov processes. A very accessible presentation embedded in a general introduction to stochastic processes can be found in Kulkarni (1995).

However, before deriving the equation for the distribution of states, some more definitions and clarifications are needed. Let Y_n denote the state of a system after the nth transition and let this state be i. Let the point in time of the nth transition be denoted by S_n.

The conditional probability of going from state i to state j in a time interval of x or shorter is given by

$$Q_{ij}\left(x\right) \equiv P\left\{Y_{n+1}=j, S_{n+1}-S_n \leq x \middle| Y_n = i\right\}.$$

Besides the fact that it might not be 1 for $x \to \infty$, $Q_{ij}\left(x\right)$ features all properties of a distribution function, compare Kulkarni (1995). Specifically, $Q_{ij}\left(x\right)$ is non-decreasing in x, $\frac{dQ_{ij}(x)}{dx} \geq 0$.

Example. A worker jumps between the two labor market states with the arrival rates being either constant or duration-dependent. As already mentioned earlier, the process is a CTMC in the first case and a SMP in the latter. Such a process is also called alternating renewal process because it alternates between these two states. The probabilities that a jump from i to j occurs in a time period shorter or equal to x is given for these alternative cases by

$$Q_{10}\left(x\right) = \left\{\begin{array}{c} 1-e^{-\lambda x} \\ 1-e^{-\int_0^x \lambda(y)dy} \end{array}\right\} \text{ for } \left\{\begin{array}{c} \text{constant } \lambda \\ \text{duration-dependent } \lambda\left(y\right) \end{array}\right., \tag{4.5}$$

$$Q_{01}\left(x\right) = \left\{\begin{array}{c} 1-e^{-\mu x} \\ 1-e^{-\int_0^x \mu(y)dy} \end{array}\right\} \text{ for } \left\{\begin{array}{c} \text{constant } \mu \\ \text{duration-dependent } \mu\left(y\right) \end{array}\right.,$$

assuming that the starting point of the time interval is 0 and the endpoint is x. Due to the homogeneity of the SMP, the probabilities and distributions only depend on the interval length x and not on where the interval is situated on the time axis.[6] The probabilities of remaining in a given state, the duration distribution, for a certain amount of time x are given in the duration-dependent case by

$$Q_{11}\left(x\right) = e^{-\int_0^x \lambda(y)dy}, \qquad Q_{00}\left(x\right) = e^{-\int_0^x \mu(y)dy}. \tag{4.6}$$

The probability that *any* transition takes place in the spell x is given by summing up the leaving probabilities for each possible state j, $Q_i\left(x\right) = \Sigma_{j \neq i} Q_{ij}\left(x\right)$, not taking into

[6]So, it holds that $Q_{ik}\left(x\right) = Q_{ik}\left(\tau | t\right)$ where $\tau = t + x$.

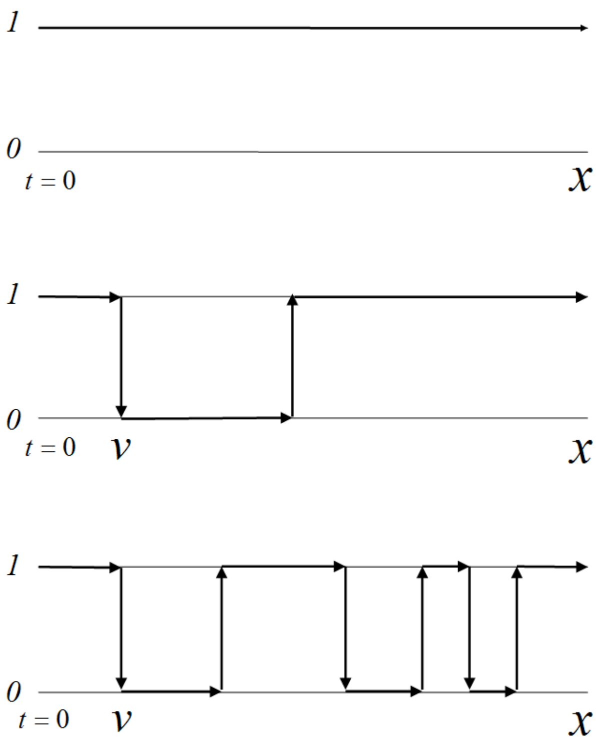

Figure 4.2: Three possible ways of starting in state 1 at $t = 0$ and ending up in state 1 a time period x later.

account transitions from i to i. In a process with two states only, this becomes

$$Q_1(x) = Q_{10}(x), \; Q_0(x) = Q_{01}(x). \tag{4.7}$$

Having done this preparation, we can now compute the probability of being in state j at x, conditioned on starting from state i today. There is a 'black box' on the way from i to j: we know that the system is in state i today and in state j a period x later, but neither do we know when this transition occurs nor whether it occurs directly or via other states. Consequently, all alternative ways of starting in i at $t = 0$ and ending up in j at x have to be taken into account. Figure 4.2 illustrates some possibilities for a continuous-time SMP with two states to start in state i and to end up in state i a time period x later.

Translating all potential transitions that could occur in that 'black box' for a multi-state process into mathematics gives the following expression:

$$\begin{aligned} p_{ij}(x) &= \delta_{ij}\left[1 - Q_i(x)\right] + \Sigma_{k \neq i} \int_0^x Q_{ik}(x - v)\, dp_{kj}(v) \\ &= \delta_{ij}\left[1 - Q_i(x)\right] + \Sigma_{k \neq i} \int_0^x dQ_{ik}(v)\, p_{kj}(x - v). \end{aligned} \tag{4.8}$$

Integral equations like equation (4.8) are Volterra equations of the first and second kind, see Polyanin and Manzhirov (1998), for example. Equation system (4.8) gives the probability that the process starting in i will be in j by x, see e.g. Kulkarni (1995) for a proof. The integral $\int_0^x Q_{ik}(x-v)\,dp_{kj}(v)$ is called the convolution of $Q_{ik}(.)$ and $p_{kj}(.)$, which is denoted by $Q_{ik} * p_{kj}(x)$. In the transition to the second line of equation (4.8), the commutativity of the convolution is used, $Q_{ik} * p_{kj}(x) = p_{kj} * Q_{ik}(x)$.

The interpretation of equation (4.8) is as follows: the first part of the right-hand side is the probability that the system, being in state i, never leaves state i until the end of the period x. In this case, $i = j$ and $\delta_{ij} = 1$, so $1 - Q_i(x)$ is the survival probability in state i. This case corresponds to the upper subfigure of figure 4.2. If $j \neq i$, then $\delta_{ij} = 0$.

The second part of the right-hand side of equation (4.8) collects all cases in which the transition from i to j occurs via another state $k \neq i$, applying the renewal argument. First, the probability that the process stays in state i for a period of length v and then passes to state k is considered, captured by $Q_{ik}(v)$. Passing to this new state k can be interpreted as a renewal of the process because the expected behavior of the process from then on is the same as whenever the process enters k. Hence, the probability that the process which is in state k at v will be in state j at x has to be taken into account, captured by $p_{kj}(x-v)$. As the transition from i to k could occur anytime between 0 and x, all possible transition times have to be covered by the integration over v. The cases, in which the transition occurred via other states is illustrated for $i = j$ in the two lower subfigures of figure 4.2.

Equation (4.8) can be rewritten, provided that $Q_{ik}(v)$ is once differentiable, as

$$p_{ij}(x) = \delta_{ij}[1 - Q_i(x)] + \Sigma_{k \neq i} \int_0^x p_{kj}(x-v) \frac{dQ_{ik}(v)}{dv} dv. \tag{4.9}$$

This equation is the origin for the following analysis based on labor market applications. As the Q_{ik} are expected to be known and differentiable in economic applications, the starting point here will be equation (4.9) rather than equation (4.8) without loss of generality.

4.3 Semi-Markov processes with two states

As stated earlier, this chapter picks the example of our labor market model from chapter 3. There are the two labor market states *unemployment* (0) and *employment* (1) and thus, four transition probabilities for the future: an unemployed / employed person can either be unemployed or employed at some future point after a spell x. Let these probabilities be denoted by $p_{ij}(x)$. Writing them out in terms of the general equation (4.9) gives

$$p_{00}(x) = 1 - Q_0(x) + \int_0^x p_{10}(x-v)\frac{dQ_{01}(v)}{dv}dv, \tag{4.10a}$$

$$p_{10}(x) = \int_0^x p_{00}(x-v)\frac{dQ_{10}(v)}{dv}dv, \tag{4.10b}$$

$$p_{11}(x) = 1 - Q_1(x) + \int_0^x p_{01}(x-v)\frac{dQ_{10}(v)}{dv}dv, \tag{4.10c}$$

$$p_{01}(x) = \int_0^x p_{11}(x-v)\frac{dQ_{01}(v)}{dv}dv. \tag{4.10d}$$

In the remainder of this section, we first discuss a special case of a SMP, namely one with constant arrival rates for both states. Since the SMP is also a CTMC in this case, the results for the probabilities from the SMP theory can be compared to the known results from CTMCs. This model is then extended in the way of chapter 3, where there are constant arrival rates in the state of employment and duration-dependent arrival rates in the state of unemployment.

4.3.1 Computing transition probabilities for constant arrival rates

Assuming a continuous-time setup, where the transition rates from one state to the other are constant, the well-known expressions for the transition probabilities of being either unemployed or employed depending on the current state can be derived. Let $p_{ij}(x)$ be the probability that a system being in state i will be in state j at a spell x later. Starting from the Chapman-Kolmogorov backward equations, a system of differential equations can be derived. The solution to this system gives the transition probabilities:

$$\begin{aligned}
p_{00}(x) &= \frac{\lambda}{\mu+\lambda} + \frac{\mu}{\mu+\lambda}e^{-[\mu+\lambda]x}, \\
p_{10}(x) &= \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda}e^{-[\mu+\lambda]x}, \\
p_{11}(x) &= \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda}e^{-[\mu+\lambda]x}, \\
p_{01}(x) &= \frac{\mu}{\mu+\lambda} - \frac{\mu}{\mu+\lambda}e^{-[\mu+\lambda]x},
\end{aligned} \tag{4.11}$$

see Ross (1996) or Kulkarni (1995), for example. In the limit as $x \to \infty$, the second terms of the probability equations approach zero. Hence, the limiting distribution does not depend on the initial distribution of states, so $p_1 = p_{01} = p_{11} = \frac{\mu}{\mu+\lambda}$ and $p_0 = p_{10} = p_{00} = \frac{\lambda}{\mu+\lambda}$. Since CTMCs are special cases of SMPs, we will now show that the transition probabilities (4.11) are special cases of the more general equations (4.10) for transition probabilities of SMPs.

First, the derivative of $Q_{01}(v)$ is prepared,

$$\frac{dQ_{01}(v)}{dv} = \mu e^{-\mu v}. \tag{4.12}$$

Inserting this into the transition probability equation (4.10) for SMPs yields

$$p_{01}(x) = \mu \int_0^x p_{11}(x - v) e^{-\mu v} dv.$$

From subsection 4.2.3, it is known that the convolution of p_{11} and Q_{01} is commutative, that means the convoluted functions and the arguments can be interchanged. Applying this gives

$$p_{01}(x) = \mu \int_0^x p_{11}(v) e^{-\mu[x-v]} dv. \tag{4.13}$$

Next, the time derivative of equation (4.13) with respect to x is computed using the Leibniz rule for integral functions, compare Wälde (2008),

$$\dot{p}_{01}(x) = \mu \left[p_{11}(x) - \mu \int_0^x p_{11}(v) e^{-\mu[x-v]} dv \right].$$

Finally, replacing the convolution by $p_{01}(x)$ from equation (4.13) yields

$$\dot{p}_{01}(x) = \mu [p_{11}(x) - p_{01}(x)] = \mu p_{11}(x) - \mu p_{01}(x). \tag{4.14}$$

This is the expected differential equation which can be derived as well from the Chapman-Kolmogorov backward equations. For the remaining three states, the corresponding differential equations can be determined in the same manner. Solving these differential equations gives the probabilities (4.11). Hence, interpreting the CTMC as a SMP with constant arrival rates leads to the same transition probabilities.

4.3.2 Computing transition probabilities for general arrival rates

From this subsection on, we use duration-dependent job arrival rates as given in our labor market model.[7]

Having non-stationary job arrival rates, the derivatives according to equation (4.5) are given by

$$\frac{dQ_{01}(v)}{dv} = e^{-\int_0^v \mu(y)dy} \frac{d}{dv} \int_0^v \mu(y)\, dy = e^{-\int_0^v \mu(y)dy} \mu(v),$$
$$\frac{dQ_{10}(v)}{dv} = e^{-\int_0^v \lambda dy} \frac{d}{dv} \int_0^v \lambda dy = e^{-\int_0^v \lambda dy} \lambda.$$

[7] Extending the model additionally by a non-stationary job-to-unemployment transition rate is also possible and would not change the general proceeding.

Together with equation (4.7) and the derivatives, the transition probabilities from equation (4.10) become

$$p_{00}(x) = e^{-\int_0^x \mu(y)dy} + \int_0^v p_{10}(x-v)\, e^{-\int_0^v \mu(y)dy} \mu(v)\, dv, \tag{4.15a}$$

$$p_{10}(x) = \int_0^x p_{00}(x-v)\, e^{-\int_0^v \lambda dy} \lambda dv, \tag{4.15b}$$

$$p_{11}(x) = e^{-\int_0^x \lambda dy} + \int_0^x p_{01}(x-v)\, e^{-\int_0^v \lambda dy} \lambda dv, \tag{4.15c}$$

$$p_{01}(x) = \int_0^x p_{11}(x-v)\, e^{-\int_0^v \mu(y)dy} \mu(v)\, dv. \tag{4.15d}$$

These four equations are central for deriving the transition probabilities of SMPs. Obviously, equations (4.15a) and (4.15b) as well as equations (4.15c) and (4.15d) are interdependent. The equation for $p_{01}(x)$ depends on $p_{11}(x-v)$ and the equation for $p_{11}(x)$, in turn, depends on $p_{01}(x-v)$. The transition probabilities $p_{11}(x)$ and $p_{01}(x)$ can be determined first and then the transition probabilities for the complementary events, $p_{10}(x)$ and $p_{00}(x)$, can be obtained immediately.[8]

One way to solve the probabilities analytically is the Laplace-Stieltjes transform, compare Kulkarni (1995). The striking fact with respect to equations (4.15a)-(4.15d) is that an analytical solution is not feasible in cases like our model because the job arrival rate has no analytical solution. Therefore, the remainder of this chapter deals with the numerical solution of the interdependent integral equations (4.15a)-(4.15d).

4.4 Numerical solution of the transition probabilities

In order to solve the transition probabilities at some point in time x numerically, at least two of the integrals in equations (4.15a)-(4.15d) have to be transformed into discrete integration problems. To this end, the interval of length x is divided into z discretization steps first. The distance between subsequent steps, the step width, is $h = x/z$ and the end point of the interval x is represented by zh. Thus, equations (4.15a) and (4.15b) become

$$p_{00}(zh) = \underbrace{e^{-\int_0^{zh} \mu(ih)d(ih)}}_{\equiv Q_{00}(zh)} + \int_0^{zh} \underbrace{e^{-\int_0^{ih} \mu(kh)d(kh)} \mu(ih)\, p_{10}(zh-ih)}_{\equiv g(ih)} d(ih) \tag{4.16}$$

[8] After having solved for two probabilities, the remaining two are the probabilities of the complementary events and can be solved by subtracting the respective probability from 1. Thus, an unemployed today can be unemployed at x, for which the probability $p_{00}(x)$ can be computed. The complementary event for the unemployed today would be occupying a job at x. As there are only the two possible states *unemployment* and *employment*, the probability for the latter is given by $p_{01}(x) = 1 - p_{00}(x)$.

and

$$p_{10}(zh) = \int_0^{zh} \underbrace{e^{-\int_0^{ih} \lambda d(kh)} \lambda p_{00}(zh - ih)}_{\equiv f(ih)} d(ih). \tag{4.17}$$

In general and independently from the numerical integration method, the approximation of the integral gets more precise the more steps are used. The drawback of having a better precision with more steps is the prolonged computing time for the integrals.

Furthermore, a numerical integration method has to be chosen in order to approximate the area beneath the function. In this section, two numerical integration methods are presented and compared in the context of the Semi-Markov transition probability problem. In subsection 4.4.1, the very basic rectangle integration method is introduced, while subsection 4.4.2 deals with the trapeze integration. These rules can be subsumed under the Newton-Cotes quadrature formulas. A general presentation can be found in Judd (1998) as well as in Schatzman and Taylor (2002).

4.4.1 Rectangle approximation

This subsection describes the numerical solution of equations (4.15a) and (4.15b) by using the rectangle approximation of integrals. As there exist several variations of the rectangle approximation, the first step is to present the general idea of computing an integral via rectangles as the basis of all variations. Then, one of the variations, the algorithm using left rectangle integration, is discussed in detail.

The general setup

As the name *rectangle approximation* already suggests, it consists of adding up the areas of rectangles beneath a function, say $\gamma(.)$. The width of every rectangle is the step-width h and the height is the function value $\gamma(ih)$ at the current position of the index i. Hence, the rectangle area is computed by $h \cdot \gamma(ih)$.

Possible variations of the rectangle method refer to the function value $\gamma(.)$, which determines the area of the first rectangle. In literature, three methods are distinguished, see Schatzman and Taylor (2002). Figure 4.3 illustrates the different methods.

As for the right rectangle method, the first rectangle is the one with height $\gamma(0)$, hence the area to the right of 0 is computed. Consequently, the rectangles from $i = 0, ..., z-1$ are added. The left rectangle method begins with the rectangle of height $\gamma(1h)$ which means that the area to the left of $1h$ is considered. In this case, the rectangles from $i = 1$,...,z are added. For the midpoint rule, the first rectangle taken is the one with height $\gamma(0.5h)$, so the function value in the middle of each interval is used. From figure 4.3

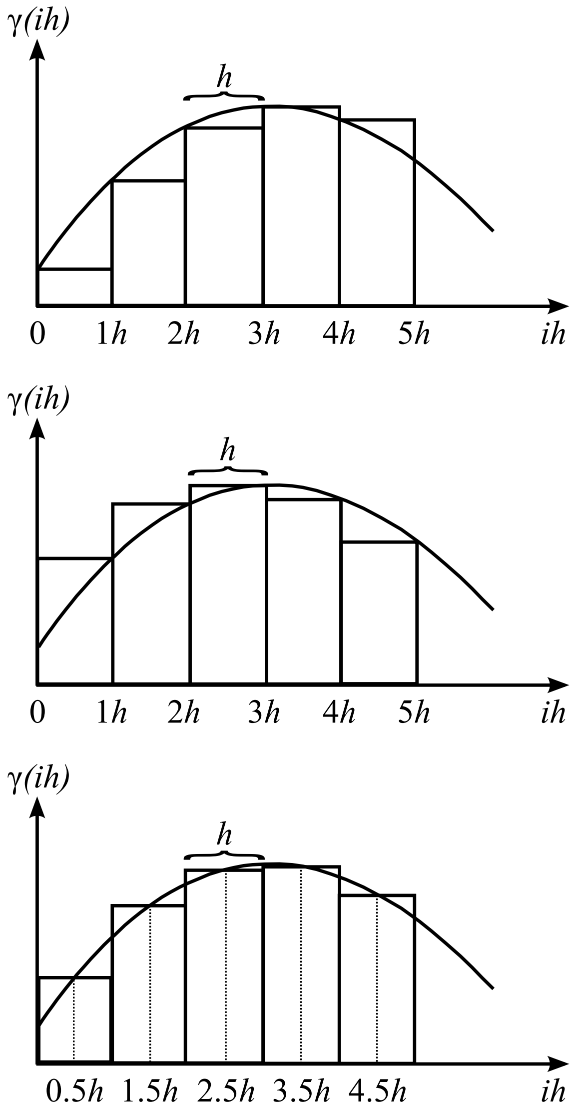

Figure 4.3: The three subfigures show the approximation of the area beneath the function via rectangles and the function values used for the rectangles. The upper figure presents the right rectangle method, the middle figure the left rectangle method, and the figure below the midpoint rule.

becomes clear why the rectangle method is a so-called *open rule*: none of the variations uses both interval endpoints, compare Judd (1998).

In the following, the left rectangle rule is discussed in detail within the Semi-Markov framework. The other two rules can be derived similarly.

Algorithm *Left Rectangles*

As mentioned above, the first function value needed for the left rectangle algorithm is the one at $i = 1$. Hence, by using the left rectangle approximation and z discretization steps the integral becomes

$$\begin{aligned} \int_0^x \gamma(v)\,dv &= h\gamma(1) + h\gamma(h) + h\gamma(2h) + ... + h\gamma(zh) \\ &= h\sum_{i=1}^{z}\gamma(ih), \end{aligned} \tag{4.18}$$

where $zh = x$ is the interval endpoint. Using the numerical integration equation (4.18), the transition probabilities for Semi-Markov processes (4.15a) and (4.15b) become

$$\begin{aligned} p_{00}(zh) &= \underbrace{e^{-h\sum_{i=1}^{z}\mu(ih)}}_{\equiv Q_{00}(zh)} + h\sum_{i=1}^{z}\underbrace{e^{-h\sum_{k=1}^{i}\mu(kh)}\mu(ih)\,p_{10}([z-i]\,h)}_{\equiv g(ih)} \\ &= Q_{00}(zh) + h\sum_{i=1}^{z} g(ih) \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} p_{10}(zh) &= h\sum_{i=1}^{z}\underbrace{e^{-h\sum_{k=1}^{i}\lambda}\lambda p_{00}([z-i]\,h)}_{\equiv f(ih)} \\ &= h\sum_{i=1}^{z} f(ih). \end{aligned} \tag{4.20}$$

Starting from the given initial values $p_{10}(0) = 0$ and $p_{00}(0) = 1$, the probabilities for any z can be computed successively, which is shown in the following algorithm.

- Initialization for $z = 0$

  The initial values $p_{00}(0)$ and $p_{10}(0)$ can be deduced intuitively. If a worker is unemployed today and no time goes by, there is no chance for him to become employed. Consequently, the probability of staying unemployed is equal to one, $p_{00}(0) = 1$. Equivalently, for an employed worker there is no risk of unemployment if no time goes by, which means $p_{10}(0) = 0$. Therefore, the initialization for the transition probabilities is given by

$$p_{00}(0) = 1,$$
$$p_{10}(0) = 0.$$

- $z = 1$

  Starting points are, like at the beginning of every step, the transition probability equations (4.19) and (4.20). Setting $z = 1$ yields

$$\begin{aligned} p_{00}(h) &= Q_{00}(h) + hg(h) \\ &= e^{-\mu(h)h} + he^{-\mu(h)h}\mu(h)\, p_{10}(0) \end{aligned}$$

  and

$$\begin{aligned} p_{10}(h) &= hf(h) \\ &= h\lambda e^{-\lambda h} p_{00}(0)\,. \end{aligned}$$

  The computation of the unknowns $p_{10}(h)$ and $p_{00}(h)$, given $p_{10}(0)$ and $p_{00}(0)$, is now straightforward.

- $z = 2$ and subsequent steps

  Evaluating equations (4.19) and (4.20) for $z = 2$ and using the definitions of $Q_{11}(ih)$, $Q_{00}(ih)$, $g(ih)$, and $f(ih)$ gives

$$\begin{aligned} p_{00}(2h) &= Q_{00}(2h) + h\sum_{i=1}^{2} g(ih) \\ &= e^{-h\sum_{i=1}^{2}\mu(ih)} + h\sum_{i=1}^{2} e^{-h\sum_{k=1}^{i}\mu(kh)}\mu(ih)\, p_{10}([2-i]\,h) \end{aligned}$$

  and

$$\begin{aligned} p_{10}(2h) &= h\sum_{i=1}^{2} f(ih) \\ &= h\lambda\sum_{i=1}^{2} e^{-h\sum_{k=1}^{i}\lambda} p_{00}([2-i]\,h)\,. \end{aligned}$$

  The further procedure for $z > 2$ is similar. In this way, the transition probabilities within an interval can be computed step by step until the probabilities for the desired point in time are reached.

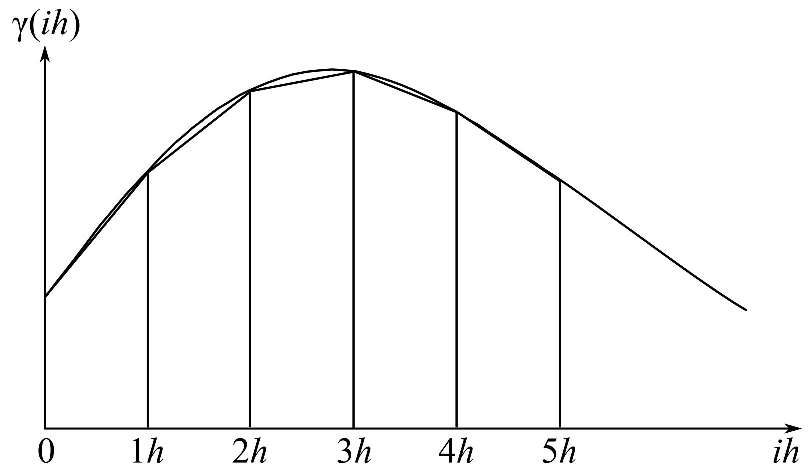

Figure 4.4: When using the trapeze rule, the area beneath the function is determined by adding up the area of the trapezes with step width h as well as side lengths $\gamma\left(ih\right)$ and $\gamma\left(\left[i-1\right]h\right)$.

4.4.2 Trapeze approximation

The second approximation rule discussed in this chapter is the trapeze rule. The integral is determined via the sum of trapeze areas beneath the function. Intuitively, the trapeze rule can be derived from the rectangle approximation by adding or subtracting triangles resulting from chords through the end points of the intervals.

The general setup

When using the trapeze approach, there is no longer a differentiation between a *right* or *left* method. As the rule uses both endpoints of the interval, it is called a *closed rule* according to Judd (1998). Figure 4.4 illustrates the trapeze approximation rule.

The trapezes taken for the approximation of the area are constructed by using the width h and the lengths $\gamma\left(\left[i-1\right]h\right)$ and $\gamma\left(ih\right)$. As for the rectangle rule, all trapeze areas in the interval are added up. Hence, an integral of a function $\gamma\left(.\right)$ becomes

$$\int_0^x \gamma\left(v\right)dv = \frac{1}{2}h\left[\gamma\left(0\right)+\gamma\left(h\right)\right]+\frac{1}{2}h\left[\gamma\left(h\right)+\gamma\left(2h\right)\right]+...+\frac{1}{2}\left[\gamma\left(\left[z-1\right]h\right)+\gamma\left(zh\right)\right].$$

Recollection results in

$$\begin{aligned}\int_0^x \gamma\left(v\right)dv &= h\left[\frac{1}{2}\gamma\left(0\right)+\gamma\left(h\right)+\gamma\left(2h\right)+...+\gamma\left(\left[z-1\right]h\right)+\frac{1}{2}\gamma\left(zh\right)\right]\\ &= \frac{1}{2}h\gamma\left(0\right)+h\sum_{i=1}^{z-1}\gamma\left(ih\right)+\frac{1}{2}h\gamma\left(zh\right). \qquad (4.21)\end{aligned}$$

Also for this method, the endpoint of the interval $x = zh$ is reached after z discretization steps and $v = ih$ is the time point of the current index position i.

In the following, the application of equation (4.21) for the computation of the transition probabilities (4.16) and (4.17) is described.

Algorithm

The general numerical integration equation (4.21) for the trapeze approximation can be used to substitute the integrals in equations (4.16) and (4.17). The former becomes

$$p_{00}(zh) = Q_{00}(zh) + \frac{1}{2}hg(0) + h\sum_{i=1}^{z-1} g(ih) + \frac{1}{2}hg(zh).$$

In addition to $p_{00}(zh)$, this equation contains a second unknown in $g(0) = \mu(0)\,p_{10}(zh)$, namely $p_{10}(zh)$. Isolating the two unknowns gives

$$p_{00}(zh) - \frac{1}{2}h\underbrace{\mu(0)\,p_{10}(zh)}_{g(0)} = Q_{00}(zh) + h\sum_{i=1}^{z-1} g(ih) + \frac{1}{2}hg(zh). \tag{4.22}$$

The full equation without the short-cut functions is written out in the appendix chapter B.2. The second equation (4.17) needs a discrete counterpart for the trapeze case, too. The procedure is equivalent, so after replacing the integrals according to equation (4.21), the probability for the transition from employment to unemployment reads

$$p_{10}(zh) = \frac{1}{2}hf(0) + h\sum_{i=1}^{z-1} f(ih) + \frac{1}{2}hf(zh).$$

This equation also has two unknowns, $p_{10}(zh)$ and $p_{00}(zh)$, because the left expression on the right-hand side, $f(0) = \lambda p_{00}(zh)$, contains the unknown $p_{00}(zh)$. Again, the final step is the isolation of both unknowns,

$$p_{10}(zh) - \frac{1}{2}h\underbrace{\lambda p_{00}(zh)}_{f(0)} = h\sum_{i=1}^{z-1} f(ih) + \frac{1}{2}hf(zh). \tag{4.23}$$

For the full version of this equation, see B.2 of the appendix. Finally, the two unknowns $p_{10}(zh)$ and $p_{00}(zh)$ from equations (4.22) and (4.23) can be determined since the $p_{10}(zh - ih)$ and $p_{00}(zh - ih)$, $i = 1, ..., z$, are given from previous calculations. In other words, by starting from $p_{10}(0) = 0$ and $p_{00}(0) = 1$, all $p(zh)$ can be solved successively. Equations (4.22) and (4.23) are the starting points of all algorithm steps, but the initialization. The algorithm steps for $z = 0$, $z = 1$, and $z = 2$ are presented in the following.

- Initialization for $z = 0$

  The initial transition probabilities from unemployment to unemployment and from employment to unemployment are given by

$$p_{00}(0) = 1$$

  and

$$p_{10}(0) = 0,$$

  respectively, for the same reason as in subsection 4.4.1 for the rectangle integration method.

- $z = 1$

  After the initialization, this is the first computation step. The basis of all computation steps are equations (4.22) and (4.23). Setting $z = 1$ in the former and using the definitions of $Q_{00}(.)$ and $g(.)$ from (4.16) yields the transition probability from unemployment to unemployment at h,

$$p_{00}(h) - \frac{1}{2}h\underbrace{\mu(0)\,p_{10}(h)}_{g(0)} = Q_{00}(h) + \frac{1}{2}h\underbrace{Q_{00}(h)\,\mu(h)\,p_{10}(0)}_{g(h)}. \tag{4.24}$$

  The transition probability from employment to unemployment at h is determined in the same manner, using $f(.)$ from equation (4.17). Setting $z = 1$ in equation (4.23) results in

$$p_{10}(h) - \frac{1}{2}h\underbrace{\lambda p_{00}(h)}_{f(0)} = \frac{1}{2}h\underbrace{e^{-\lambda h}\lambda p_{00}(0)}_{f(h)}. \tag{4.25}$$

  Equations (4.24) and (4.25) are the first two equations with the first two unknowns $p_{00}(h)$ and $p_{10}(h)$. The solution is now straightforward.

- $z = 2$ and subsequent steps

  The next step is to go on with $z = 2$ and to compute $p_{00}(2h)$ as well as $p_{10}(2h)$ given the results from all previous steps. Equations (4.22) and (4.23) become

$$\begin{aligned} p_{00}(2h) - \frac{1}{2}h\underbrace{\mu(0)\,p_{10}(2h)}_{g(0)} = Q_{00}(2h) + h\underbrace{Q_{00}(h)\,\mu(h)\,p_{10}(h)}_{g(h)} \\ + \frac{1}{2}h\underbrace{Q_{00}(2h)\,\mu(2h)\,p_{10}(0)}_{g(2h)} \end{aligned}$$

and

$$p_{10}(2h) - \frac{1}{2}h\underbrace{\lambda p_{00}(2h)}_{f(0)} = h\underbrace{e^{-\lambda h}\lambda p_{00}(h)}_{f(h)} + \frac{1}{2}h\underbrace{e^{-\lambda 2h}\lambda p_{00}(0)}_{f(2h)},$$

respectively.

The only two unknowns in step 2 are $p_{10}(2h)$ and $p_{00}(2h)$ on the left-hand side because $p_{10}(0)$ and $p_{00}(0)$ are known from the initialization and $p_{10}(h)$ and $p_{00}(h)$ from the first step. So also this equation system can be solved for the probabilities at $x = 2h$.

The proceeding for the subsequent steps with $z = 3, ...$ equivalently starts from equations (4.22) and (4.23). The mechanism is always the same: the $p_{00}(zh)$ and $p_{10}(zh)$ are calculated using the $p_{00}(zh - ih)$ and $p_{10}(zh - ih)$, $i = 1, ..., z$, from the previous steps.

After the theoretical description of possible numerical solution methods, the next section shows the computational results for specific numerical examples.

4.5 Numerical results

Having learned two alternatives of determining transition probabilities in the previous section, this section focuses on how both solutions perform when applying them to specific labor market models.[9]

First, the methods of numerical integration discussed in chapter 4.4, the rectangle and the trapeze method, are compared to the analytically computable transition probabilities in the case of constant arrival rates as given by equations (4.11). In general, it is clear that the trapeze method will perform better than the rectangle method when using the same step width and step number. However, an important question is how much better the trapeze method is when employing it for the solution of our labor market model, considering that the trapeze method is more complex and will need more computation time, consequently. Furthermore, the limiting distribution as derived by equation (4.4) will be tested. Thus, the analytical solution serves as a benchmark for the numerical methods in the case of constant transition rates.

Second, the probabilities for duration-dependent arrival rates are computed with both numerical methods. As there is no longer an analytical solution available in cases like our economic model of chapter 3, the two solutions can only be analyzed independently.

[9]The algorithm of the solution procedure is set up in Matlab. The code is available on the enclosed CD.

However, the limiting distribution can be computed for Semi-Markov processes and, in this way, at least the convergence of both numerical solutions can be evaluated.

4.5.1 Constant arrival rates - convergence to the analytical solution

In order to test the convergence of the transition probabilities computed via the numerical algorithms, constant arrival rates are used. In this special case, the SMP is a CTMC, for which the analytical solution of the transition probabilities is known, see equations (4.11) in subsection 4.3.1. The parameters used for this analysis are taken from Shimer (2005). The monthly values are $\mu = 0.45$ for the job arrival rate and $\lambda = 0.034$ for the job separation rate. The interval endpoint is $x = 500$ months. The limiting distribution is then given by $p_1^A = \frac{\mu}{\mu+\lambda} = 0.93$ and $p_0^A = \frac{\lambda}{\mu+\lambda} = 0.07$ according to subsection 4.3.1.

- Comparison of graphs

Figure 4.5 shows the evolution of the transition probabilities for the analytical solution compared to the numerical solution of the *rectangle* method. Each subfigure presents the probabilities for different step numbers. The probability for the transition from initial unemployment to unemployment is 1 for $t = 0$, the probability for the transition from initial employment to unemployment is 0 as $t = 0$.[10] The analytical solution reaches the limiting distribution at about $t = 20$ months and the two analytical curves can no longer be distinguished from then on. The rectangle probabilities do not seem to converge at all for the displayed step numbers. For 250 steps, the numerical solution using the rectangle method clearly underestimates the probabilities for $t \geq 25$, see the upper subfigure. At the endpoint of the figure at $t = 150$, the numerically approximated probabilities are nearly zero. For $2,000$ steps, there is still underestimation of the analytical probabilities, but the magnitude decreases and the difference between the two computation methods at $t = 150$ is much smaller than before.

[10] See initialization step for $z = 0$ in the previous section for the explanation.

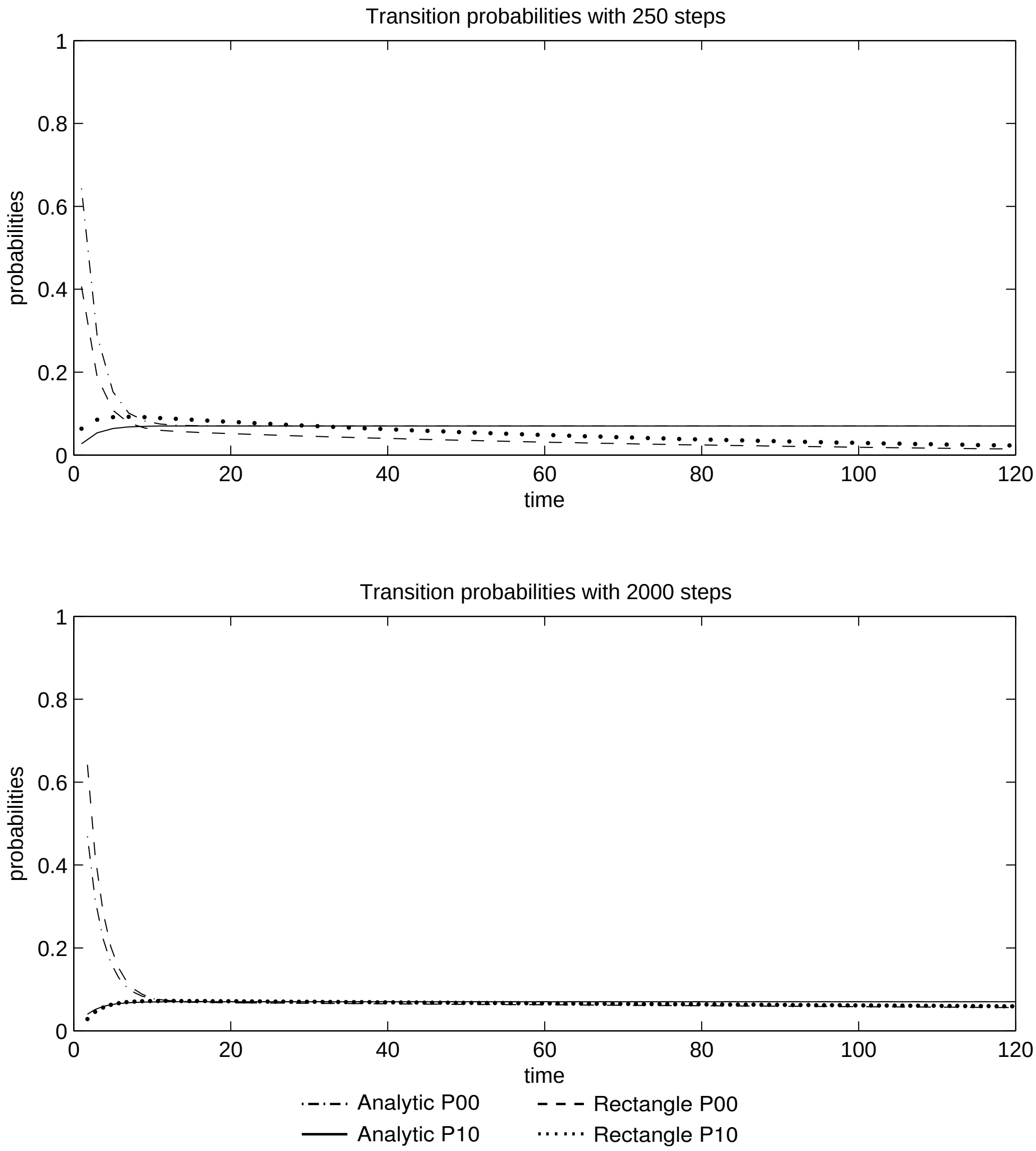

Figure 4.5: Transition probabilities over time for the analytical solution and the rectangle method. The upper figure shows the solution for 250 steps and the figure at the bottom for $2,000$ steps.

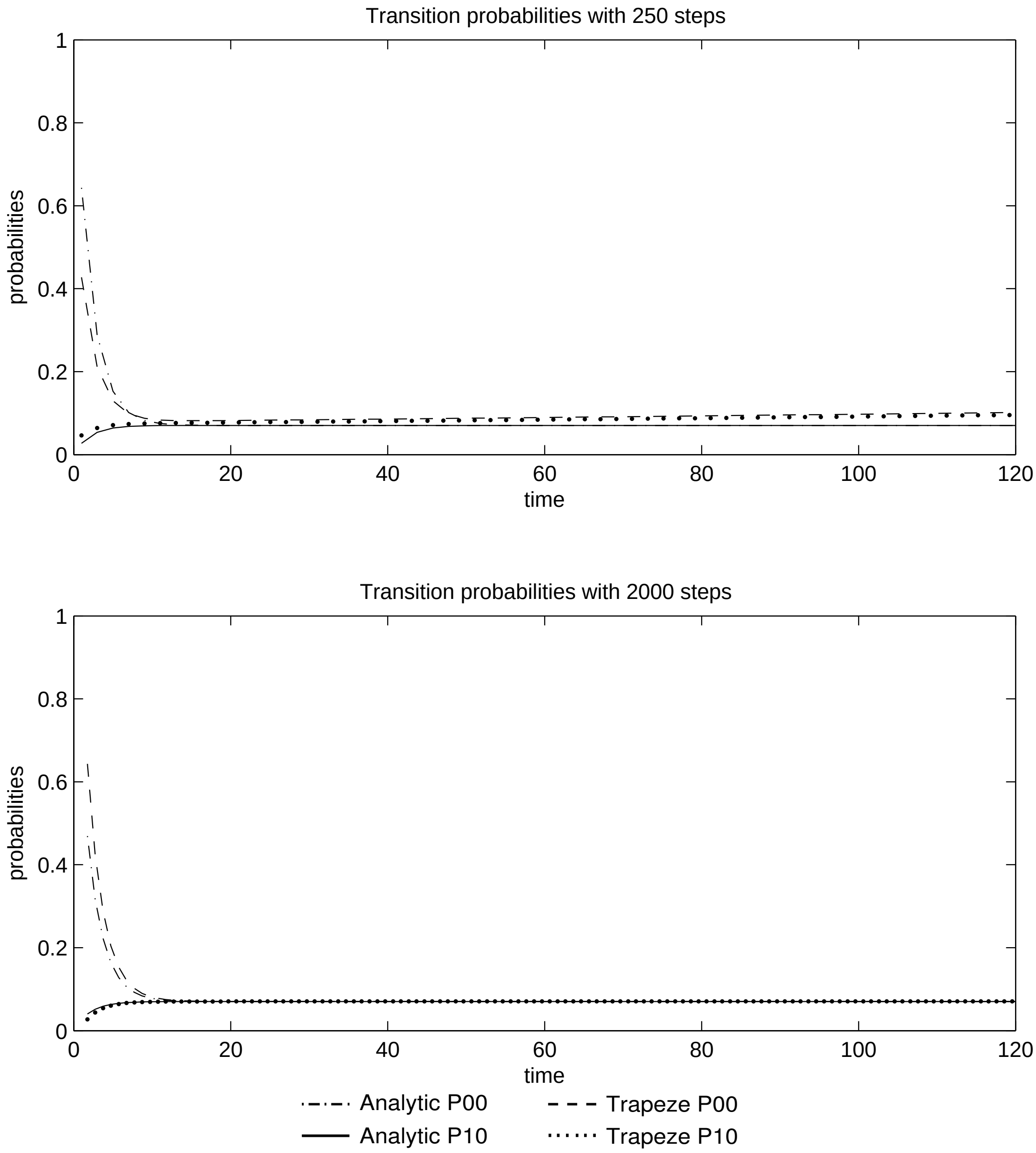

Figure 4.6: Transition probabilities over time for the analytical solution and the trapeze method. The upper figure shows the solution for 250 steps and the figure at the bottom for $2,000$ steps.

Figure 4.6 shows the transition probabilities for the analytical solution compared to the numerical solution of the *trapeze* approximation, again for different step numbers. Convergence is much better than for the rectangle solution. Already for $2,000$ steps, the trapeze probabilities approach the same limiting value as the analytical solution. As before, the probability for the transition from initial unemployment to unemployment at $t = 0$ is 1, whereas the probability for the transition from initial employment to unemployment at $t = 0$ is 0. The upper subfigure in figure 4.6 shows the curves for 250 steps. After the first 20 months, there is a monotonically increasing overestimation. The trapeze

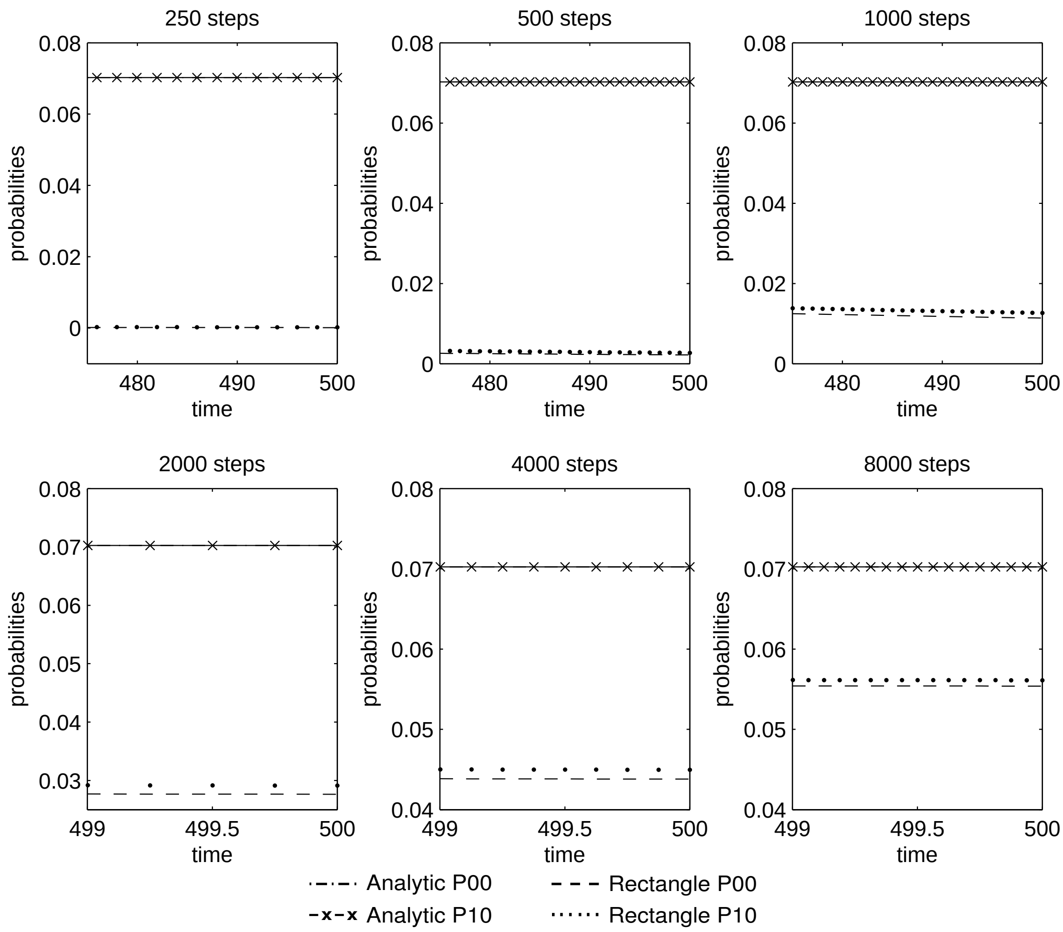

Figure 4.7: Transition probabilities for the analytical solution and the rectangle solution as $t \to 500$ for different step numbers. The upper figures show the interval $[475, 500]$, the bottom figures show the interval $[499, 500]$.

solution is obviously still much better than the rectangle method described above. The lower subfigure shows the probability evolution for $2,000$ steps. The improvement from 250 steps to $2,000$ steps is large, especially from $t = 20$ onwards. For this step number, there is nearly no difference between the curves of the analytical solutions and the curves of the numerical trapeze solutions visible. After this overview of the probability evolution, some more detailed figures on the behavior as $t \to 500$ will be shown.

Figure 4.7 shows the probabilities for the transitions from unemployment to unemployment and from employment to unemployment both for the analytical solution and the *rectangle* approximation zoomed in near the endpoint of the interval. Now, the range of the underestimation of the analytical solution by the rectangle approximation becomes better visible. Clearly, the numerical solution approaches the analytical solution as the step number increases with the errors getting smaller for increasing step numbers.

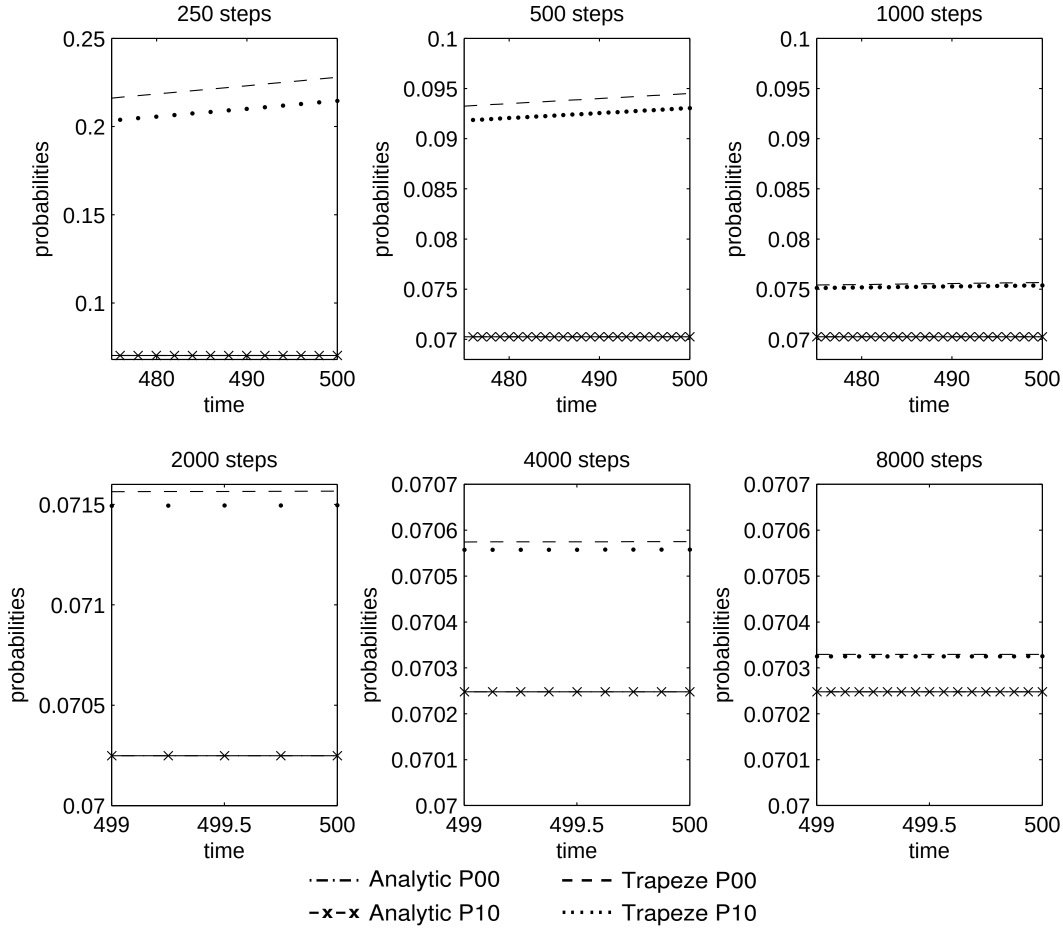

Figure 4.8: Transition probabilities for the analytical solution and the trapeze solution as $t \to 500$, again for different step numbers. The upper figures show the interval $[475, 500]$, the bottom figures show the interval $[499, 500]$.

Figure 4.8 shows the corresponding probabilities for the *trapeze* approximation compared to the analytical solution. Also these figures verify that, for a bigger step number, the numerical transition probabilities perform better as approximations of the analytical solution. Furthermore, it becomes obvious that the trapeze approximation method overestimates the analytical solution, but, unlike for the rectangle probabilities, already the solutions for $2,000$ steps perform quite good. Having an equivalently good approximation in the rectangle case would require 8,000 or more computation steps.

- Comparison by computational results

Table 4.1 and table 4.2 present the computational results for different step numbers and the three methods (analytical, rectangle, trapeze). The solutions and errors of both numerical integration methods are compared to the analytical solution at different points of the interval. While in the former table the results for the transition probabilities from

		250 steps		500 steps		2,000 steps	
		Value	Error	Value	Error	Value	Error
	p_{00}^{A}	0.070	-	0.070	-	0.070	-
1/5	p_{00}^{R}	0.019	-0.051	0.035	-0.035	0.058	-0.018
	p_{00}^{T}	0.097	+0.027	0.076	+0.006	0.071	+0.001
	p_{00}^{A}	0.070	-	0.070	-	0.070	-
1/2	p_{00}^{R}	0.003	-0.067	0.012	-0.058	0.044	-0.026
	p_{00}^{T}	0.134	+0.057	0.083	+0.013	0.071	+0.001
	p_{00}^{A}	0.070	-	0.070	-	0.070	-
End	P_{00}^{R}	0.000	-0.07	0.002	-0.068	0.028	-0.042
	p_{00}^{T}	0.228	+0.158	0.095	+0.025	0.072	+0.002
		4,000 steps		8,000 steps		16,000 steps	
		Value	Error	Value	Error	Value	Error
	p_{00}^{A}	0.070	-	0.070	-	0.070	-
1/5	p_{00}^{R}	0.064	-0.006	0.067	-0.003	0.069	-0.001
	p_{00}^{T}	0.070	-	0.070	-	0.070	-
	p_{00}^{A}	0.070	-	0.070	-	0.070	-
1/2	p_{00}^{R}	0.055	-0.015	0.062	-0.008	0.066	-0.004
	p_{00}^{T}	0.070	-	0.070	-	0.070	-
	p_{00}^{A}	0.070	-	0.070	-	0.070	-
End	P_{00}^{R}	0.044	-0.026	0.055	-0.015	0.062	-0.008
	p_{00}^{T}	0.071	+0.001	0.070	-	0.070	-

Table 4.1: Probabilities for the transition from unemployment to unemployment $p_{00}(.)$ by t_i, where $t_1 = 1/5 \cdot x = 100$, $t_2 = 1/2 \cdot x = 250$, and $t_3 = x = 500$.

unemployment to unemployment, $p_{00}(t)$, are recorded, the latter shows the transition probabilities from employment to unemployment, $p_{10}(t)$.[11]

The columns present the probabilities for different step numbers, the rows show the probabilities for the three computation methods analytical, rectangle, and trapeze for different points in the interval $[0, 500]$. First, the probabilities at $1/5$ of the interval, $t_1 = 100$, then the probabilities after half of the interval at $t_2 = 250$, and finally, the probabilities at the endpoint $x = 500$ are compared for the three methods.

Table 4.1 shows the probabilities for the transition from initial unemployment to unemployment, $p_{00}(.)$. For 250 (500) steps and after $1/5$ of the time, the rectangle solution underestimates the analytical solution in a range of 73% (50%), whereas the trapeze solution overestimates the analytical solution in a range of 39% (8.6%). So at $t_1 = 100$, the

[11]Note that the probabilities for the complementary events can easily be determined via $p_{11}(t) = 1 - p_{10}(t)$ and $p_{01}(t) = 1 - p_{00}(t)$, respectively.

trapeze solution performs much better than the rectangle solution. With increasing step numbers, both approximated probabilities continuously get better at $t_1 = 100$ with the trapeze solution being much better than the rectangle solution. Already at $4,000$ steps, the deviation of the trapeze probability from the analytical one is 0% within the chosen accuracy of three decimal places. At the interval endpoint $x = 500$ with 250 (500) steps, both probabilities are very bad estimates for the analytical probability with an error of 100% (33%) or higher. As expected, the error decreases with increasing step numbers, so at the interval endpoint with $8,000$ steps, there is no longer a significant error for the trapeze solution. The best result for the rectangle solution at the endpoint $x = 500$ with $16,000$ steps still delivers an error of 10%, which is disproportionatly high given the required amount of computation effort. So in order to get results for the rectangle method, which are equally good like for the trapeze method with $2,000$ steps requires $16,000$ steps or more.
In the analytic case, convergence is reached at about 20 months. Using adequate step numbers, it also takes both approximation methods around 20 months until convergence to the limiting distribution.

Table 4.2 shows the probabilities for the transition from initial employment to unemployment, $p_{10}(.)$. For 250 (500) steps and after $1/5$ of the time, the underestimation by the rectangle solution is not as big as for the corresponding $p_{00}(1/5)$ probabilities with the error being about 57% (39%). The trapeze solution overestimates the analytical solution in a range of 31% (7%). So at $t_1 = 100$, the trapeze solution again performs much better than the rectangle solution. With increasing step numbers, both approximated probabilities continuously get better at $t_1 = 100$ as it has already been the case for the $p_{00}(.)$ probabilities. This holds for all considered points of time in the interval: starting from the unacceptable 250 and 500 step cases, the results at all observed interval points get better, the more steps are used for the calculation. Again, the results for the trapeze method and $2,000$ steps are better than the results for the rectangle method with $16,000$ steps.

- Convergence with respect to the limiting distribution

The limiting distribution of the SMP can be determined using equation (4.4). However, the integrals cannot be evaluated analytically as soon as there is no analytic solution for $\mu(.)$. Hence, also for the limiting distribution, the accuracy of the different numerical integration methods is evaluated. The analytical limiting distribution values are $p_1^A = \frac{\mu}{\mu+\lambda} = 0.93$ and $p_0^A = \frac{\lambda}{\mu+\lambda} = 0.07$ according to equation (4.1). For both integration methods, the computed values of the limiting distribution can be taken from

		500 steps		1,000 steps		2,000 steps	
		Value	Error	Value	Error	Value	Error
	p_{10}^A	0.070	-	0.070	-	0.070	-
1/5	p_{10}^R	0.030	-0.04	0.043	-0.027	0.061	-0.009
	p_{10}^T	0.092	+0.022	0.075	+0.005	0.071	+0.001
	p_{10}^A	0.070	-	0.070	-	0.070	-
1/2	p_{10}^R	0.004	-0.066	0.015	-0.055	0.046	-0.024
	p_{10}^T	0.126	+0.056	0.081	+0.011	0.071	+0.001
	p_{10}^A	0.070	-	0.070	-	0.070	-
End	p_{10}^R	0.000	-0.07	0.003	-0.067	0.029	-0.041
	p_{10}^T	0.215	+0.145	0.093	+0.023	0.071	+0.001
		4,000 steps		8,000 steps		16,000 steps	
		Value	Error	Value	Error	Value	Error
	p_{10}^A	0.070	-	0.070	-	0.070	-
1/5	p_{10}^R	0.066	-0.004	0.068	-0.002	0.069	-0.001
	p_{10}^T	0.070	-	0.070	-	0.070	-
	p_{10}^A	0.070	-	0.070	-	0.070	-
1/2	p_{10}^R	0.057	-0.013	0.063	-0.007	0.067	-0.003
	p_{10}^T	0.070	-	0.070	-	0.070	-
	p_{10}^A	0.070	-	0.070	-	0.070	-
End	p_{10}^R	0.045	-0.025	0.056	-0.014	0.063	-0.007
	p_{10}^T	0.071	+0.001	0.070	-	0.070	-

Table 4.2: Probabilities for the transition from employment to unemployment $p_{10}(.)$ by t_i, where $t_1 = 1/5 \cdot x = 100$, $t_2 = 1/2 \cdot x = 250$, and $t_3 = x = 500$.

	p_0^R	p_0^T
250 steps	0.046	0.075
500 steps	0.057	0.071
1,000 steps	0.064	0.071
2,000 steps	0.067	0.070
4,000 steps	0.069	0.070
8,000 steps	0.069	0.070
16,000 steps	0.070	0.070
$p_0^A = 0.070$		

Table 4.3: Limiting probabilities p_0, computed via the two numerical integration methods at different step numbers. The last line shows the analytical value. The remaining probability of the distribution can be calculated by $p_1 = 1 - p_0$.

table 4.3. Besides the numerical integration method, there is now a second source of inexactness, namely the approximation of infinity by 500. However, as the trapeze method delivers very good estimates of the limiting distribution already for smaller time values, approximating infinity by 500 appears to be reasonable when computing the expectation. All in all, the trapeze method is also for the limiting distribution precise enough given our purpose: using $2,000$ steps already results in an error of 0% for three decimals preciseness, while the rectangle method still needs $16,000$ steps.

In summary, the trapeze solution performs much better as an approximation for the analytically computed CTMC transition probabilities and the limiting distribution than the rectangle method for the given labor market framework. This better exactness comes along with an extended computation effort since the trapeze method is more complex. However, the increased computation effort due to the higher complexity can be reduced again: the trapeze method requires less steps in order to reach a given accuracy. While for our purposes, $2,000$ steps prove to be exact enough when using the trapeze method, we would need $16,000$ steps or more to reach acceptable results for the rectangle method. Altogether, the choice of the integration method should be made depending on the complexity and the scope of the underlying project.

4.5.2 Duration-dependent arrival rates

In this subsection, the transition probabilities in a setup with duration-dependent arrival rates for jobs $\mu(.)$ and constant separation rates λ are computed. The $\mu(.)$ are taken from

	$p_{00}^{R}(.)$	$p_{00}^{T}(.)$	$p_{10}^{R}(.)$	$p_{10}^{T}(.)$
100	0.178	0.179	0.163	0.163
250	0.165	0.167	0.166	0.167
500	0.162	0.167	0.163	0.167

Table 4.4: Transition probabilities for duration-dependent transition rates at different points in time for $2,000$ steps.

our labor market model of chapter 3. The parameters used are $\lambda = 0.0098$, $2,000$ steps, and again, 500 as the interval endpoint. It is no longer possible to compare the numerical solutions to analytical solutions because an analytical solution is no longer available. However, the evolution of both methods can still be considered and discussed, as well as the convergence to the limiting probabilities.

- Evolution of $p_{ij}(t)$ for increasing t using rectangle and trapeze approximation

Figure 4.9 shows the evolution of the transition probabilities over time using $2,000$ steps. The trapeze approximation approaches a limiting value of about 0.167, while the rectangle probabilities slightly keep decreasing. Table 4.4 shows some selected values.

As there is no longer an analytical benchmark for the probabilities, the next step is to compute the limiting distributions by the two numerical integration methods.

- Convergence with respect to the limiting distribution

Using equation (4.4) with the two numerical integration methods and approximating infinity by 500 gives estimates of the limiting distribution for each method. For the rectangle method, it is given by

$$p_0^R = 0.1683\ ,\ p_1^R = 0.8317, \tag{4.26}$$

while the trapeze method yields

$$p_0^T = 0.1684\ ,\ p_1^T = 0.8316. \tag{4.27}$$

These values are quite similar and they can be compared to the limiting values from above. For $2,000$ steps, the trapeze solution performs again better, as can be seen from table 4.4. The trapeze solution at $t = 500$ of about 0.167 is nearer to both the trapeze limit of 0.1684 and the rectangle limit of 0.1683 than the rectangle solution at $t = 500$. This result is in accordance with the findings from the previous subsection.

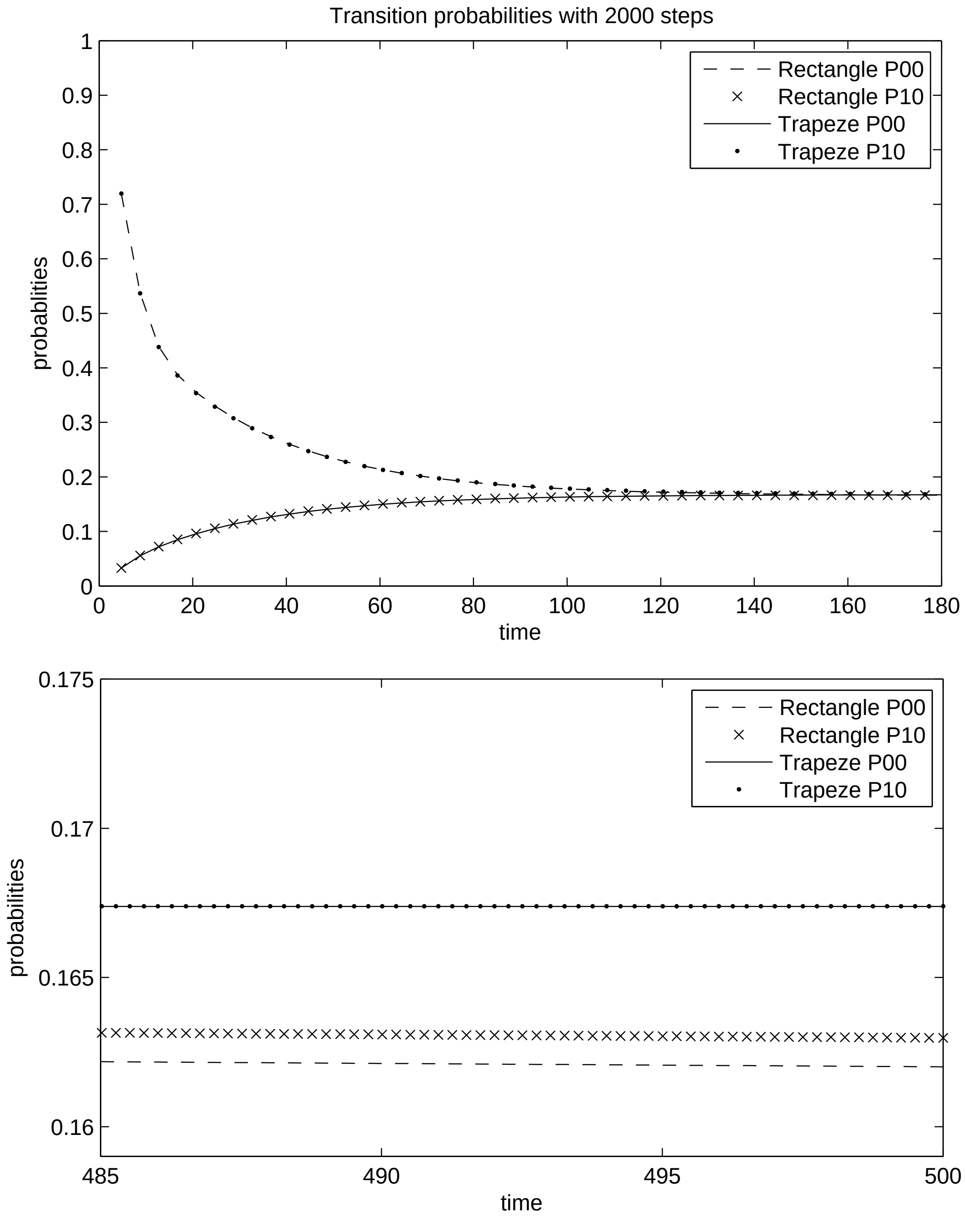

Figure 4.9: Transition probabilities of the labor market model with duration-dependent job arrival rates in the interval $[0, 180]$ (upper subfigure) and in the interval $[485, 500]$ (lower subfigure) for $2,000$ steps.

4.6 Conclusion

The use of Semi-Markov processes allows a more realistic description of behavior or states in economic modeling. In labor market theory, duration-dependent transition rates account for microeconomic reactions of individuals over the unemployment spell due to incentive effects of non-stationary benefit schemes, for example. This chapter is devoted to the application of Semi-Markov processes in this area, especially with respect to the derivation of the conditional and unconditional distribution of labor market states. To this end, a basic introduction to Semi-Markov theory is given first. Then, we show how to determine the transition probabilities between labor market states and the limiting distribution of states by means of the labor market model from chapter 3, where a Semi-Markov structure appears in the setup. Since the calculation requires the application of numerical integration methods, two selected methods, the rectangle and the trapeze approximation, are introduced and compared with respect to the accuracy of their numerical results for different step numbers.

Based on a specific labor market example and with constant arrival rates, a step width of about $1/4$ appears to be accurate enough for precise results when using the trapeze rule. For the rectangle method, results are equally acceptable at a step width not more than $1/32$. Regarding the limiting distribution, the trapeze method delivers a very good approximation already at step width $1/4$ with the error being 0% within the chosen preciseness. Also here, the rectangle method requires a much finer step width.

For duration-dependent arrival rates, the transition rates are taken from our labor market model of chapter 3. Also in this case, the transition probabilities of both numerical integration methods approach a limiting value. Again, the trapeze method for the transition probabilities at $x = 500$ converge better to the numerical limiting probabilities computed by both the trapeze method and the rectangle method.

Altogether, the trapeze method is a much more precise method at much smaller step numbers and, therefore, provides higher computation efficiency. Hence, for the transition probabilities of our labor market model it is reasonable to prefer this slightly complexer method over the rectangle method while using less steps.