9 Referees' appendix to "Endogenous growth cycles" by Klaus Wälde

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9.1 A vintage capital structure - deriving (10)

Vintage-specific technologies are given by (1). Labour mobility implies equality of nominal wages for all vintages j, $w_j = w_0 \; \forall j$. The wage rate implied by vintage j is given by $w_j = p_c (1-\alpha) \left(\frac{K_j}{A^j L_j}\right)^\alpha A^j$. The wage rate of vintage 0 is $w_0 = p_c (1-\alpha) \left(\frac{K_0}{A^0 L_0}\right)^\alpha A^0$. Equality of wages for vintages 0 and j implies labour allocation to vintage j relative to vintage 0 of

$$w_0 = w_j \Leftrightarrow \left(\frac{K_0}{A^0 L_0}\right)^\alpha A^0 = \left(\frac{K_j}{A^j L_j}\right)^\alpha A^j \Leftrightarrow \frac{K_0}{A^0 L_0} = A^{\frac{j}{\alpha}} \frac{K_j}{A^j L_j}$$
$$\Leftrightarrow L_j = A^{\frac{j}{\alpha}} \frac{K_j}{A^j} \frac{A^0}{K_0} L_0 = A^{j\frac{1-\alpha}{\alpha}} \frac{K_j}{K_0} L_0. \tag{55}$$

Inserting into the labour market clearing condition $\Sigma_{j=0}^q L_j = L$ yields

$$L_0 + A^{\frac{1-\alpha}{\alpha}} \frac{K_1}{K_0} L_0 + ... + A^{q\frac{1-\alpha}{\alpha}} \frac{K_q}{K_0} L_0 = L \Leftrightarrow$$
$$\frac{L_0}{K_0}\left(K_0 + A^{\frac{1-\alpha}{\alpha}} K_1 + ... + A^{q\frac{1-\alpha}{\alpha}} K_q\right) = L \Leftrightarrow L_0 = \frac{K_0}{K} L, \tag{56}$$

where $K \equiv K_0 + A^{\frac{1-\alpha}{\alpha}} K_1 + ... + A^{q\frac{1-\alpha}{\alpha}} K_q \equiv K_0 + BK_1 + ... + B^q K_q$. Inserting (56) in (55) gives labour allocation to vintage j, $L_j = \frac{A^{j\frac{1-\alpha}{\alpha}} K_j}{K} L$. Output of vintage j is therefore $Y_j = K_j^\alpha (A^j L_j)^{1-\alpha} = K_j^\alpha \left(\frac{A^{\frac{j}{\alpha}} K_j}{K} L\right)^{1-\alpha} = K_j \left(A^{\frac{j}{\alpha}} \frac{L}{K}\right)^{1-\alpha}$. Total output is then given by $Y = Y_0 + ... + Y_q = \left(K_0 + K_1 A^{\frac{1-\alpha}{\alpha}} + ... + K_q A^{q\frac{1-\alpha}{\alpha}}\right) \left(\frac{L}{K}\right)^{1-\alpha} = K^\alpha L^{1-\alpha}$.

9.2 The budget constraint (20)

Real wealth a of households is given by the sum of the number k_j of units of capital of vintage j held by the household times their real price v_j/p_Y,

$$a = \Sigma_{j=0}^{q+1} k_j \frac{v_j}{p_Y}. \tag{57}$$

For reasons that will become clear in a moment, the sum extends from 0 to $q+1$, though the most advanced vintage is vintage q and household therefore can not own any capital of vintage $q+1$. Households trade only capital goods of the most recent vintage. The allocation of older capital goods is fixed (in equilibrium, households would be indifferent about trading old capital goods). Capital held by households therefore follows for old vintages j

$$dk_j = -\delta k_j dt, \quad \forall j < q, \tag{58}$$

for the most recent one

$$dk_q = \left(\frac{\Sigma_{j=0}^{q+1} w_j^K k_j + p_c w - p_R i - p_c c}{v_q} - \delta k_q \right) dt, \tag{59}$$

and for the next vintage $q + 1$

$$dk_{q+1} = \kappa \frac{i}{R} dq. \tag{60}$$

The capital stock k_q in (59) of a household increases in a deterministic fashion when the difference between actual nominal income and spending, $\Sigma_{j=0}^{q+1} w_j^K k_j + p_c w - p_R i - p_c c$, divided by the price v_q of an installed or the price p_I of a new unit of capital exceeds losses δk_q of capital due to depreciation. Capital income $\Sigma_{j=0}^{q+1} w_j^K k_j$ of households is given by nominal factor rewards w_j^K for capital from (13) times the amount of capital k_j, summed up over all vintages.

Equation (60) shows that in the case of a successful R&D project, i.e. when $dq = 1$, the household obtains the share i/R, real individual investment i relative to real total investment R, of total payoffs κ. A successful research project therefore increases the capital stock of vintage $q + 1$ held by the household from 0 to $\kappa i/R$. After that, equation (59) applies to vintage $q + 1$.

The price of a vintage j in terms of the numeraire good is given by (15) with (3). Hence, letting vintage prices evolve in all generality as

$$d\frac{v_j}{p_Y} = \alpha_j \frac{v_j}{p_Y} dt - \gamma_s \frac{v_j}{p_Y} dq, \tag{61}$$

we know that the deterministic change of the real price v_j/p_Y must be zero, $\alpha_j = 0 \, \forall j = 0...q$. When research is successful, the price of a unit of a given vintage j in terms of the numeraire good drops as then, by (15) and (3), $\tilde{v}_j/\tilde{p}_Y = B^{j-(q+1)}$.[22] Hence, as $d(v_j/p_Y) = \tilde{v}_j/\tilde{p}_Y - v_j/p_Y$, we have $d(v_j/p_Y) = B^{j-(q+1)} - B^{j-q}$. As a consequence and with (61),

$$-\gamma_s = \frac{d(v_j/p_Y)}{v_j/p_Y} = \frac{B^{j-(q+1)} - B^{j-q}}{B^{j-q}} = \frac{1-B}{B} < 0,$$

which is identical for all vintages $j \leq q$. Real vintage prices (61) therefore evolve according to

$$d\frac{v_j}{p_Y} = -\frac{B-1}{B} \frac{v_j}{p_Y} dq \quad \forall j \leq q. \tag{62}$$

This equation reflects the economic depreciation of old vintages relative to the numeraire good when a new vintage has been developed.

[22] A tilde ($\tilde{}$) denotes the value of a quantity immediately after successful research.

We can now derive the budget constraint by computing the differential $da = \Sigma_{j=0}^{q+1} d\left(\frac{v_j}{p_Y}k_j\right)$. For all vintages $0 < j < q$, we obtain with (58) and (62) and using Ito's Lemma

$$\begin{aligned} d\left(\frac{v_j}{p_Y}k_j\right) &= -\frac{v_j}{p_Y}\delta k_j dt + \left[\left(\frac{v_j}{p_Y} - \frac{B-1}{B}\frac{v_j}{p_Y}\right)k_j - \frac{v_j}{p_Y}k_j\right]dq \\ &= -\delta\frac{v_j}{p_Y}k_j dt - \frac{B-1}{B}\frac{v_j}{p_Y}k_j dq \quad \forall j = 0...q-1. \end{aligned}$$

For the currently most advanced vintage q, we use (59) and (62) to obtain

$$\begin{aligned} d\left(\frac{v_q}{p_Y}k_q\right) &= \frac{v_q}{p_Y}\left(\frac{\Sigma_{j=0}^{q+1} w_j^K k_j + p_c w - p_R i - p_c c}{v_q} - \delta k_q\right)dt \\ &+ \left[\left(\frac{v_q}{p_Y} - \frac{B-1}{B}\frac{v_q}{p_Y}\right)k_q - \frac{v_q}{p_Y}k_q\right]dq \\ &= \left(\Sigma_{j=0}^{q+1}\frac{w_j^K}{p_Y}k_j + w - i - c - \delta\frac{v_q}{p_Y}k_q\right)dt - \frac{B-1}{B}\frac{v_q}{p_Y}k_q dq. \end{aligned}$$

For the next vintage $q+1$ to come, from (60) and with a real price $\tilde{v}_{q+1}/\tilde{p}_Y$ for the prototype after successful R&D, i.e. only when the good κ exists,

$$d\left(\frac{v_{q+1}}{p_Y}k_{q+1}\right) = \left(\frac{\tilde{v}_{q+1}}{\tilde{p}_Y}\kappa\frac{i}{R} - 0\right)dq = \kappa\frac{i}{R}dq. \tag{63}$$

The real price equals unity, $\tilde{v}_{q+1}/\tilde{p}_Y = 1$ from (15). Hence, κ stands for the number of consumption goods that can be exchanged for the prototype. This is in accordance with the definition of real wealth in (57) which also is the number of consumption goods that can be exchanged for a.

Summarizing, we obtain[23]

$$da = \Sigma_{j=0}^{q+1} d\left(k_j\frac{v_j}{p_Y}\right)$$

$$= \Sigma_{j=0}^{q-1}\left(-\delta\frac{v_j}{p_Y}k_j dt - \frac{B-1}{B}\frac{v_j}{p_Y}k_j dq\right) + \left(\Sigma_{j=0}^{q+1}\frac{w_j^K}{p_Y}k_j + w - i - c - \delta\frac{v_q}{p_Y}k_q\right)dt$$
$$- \frac{B-1}{B}\frac{v_q}{p_Y}k_q dq + \kappa\frac{i}{R}dq$$

$$= \Sigma_{j=0}^{q}\left(-\delta\frac{v_j}{p_Y}k_j dt - \frac{B-1}{B}\frac{v_j}{p_Y}k_j dq\right) + \left(\Sigma_{j=0}^{q+1}\frac{w_j^K}{p_Y}k_j + w - i - c\right)dt + \kappa\frac{i}{R}dq$$
$$= \left(\frac{p_c}{p_Y}\frac{\partial Y}{\partial K}\Sigma_{j=0}^{q+1}B^j k_j - \delta a + w - i - c\right)dt + \left(\kappa\frac{i}{R} - \frac{B-1}{B}a\right)dq.$$

[23] Here we need assets a to equal the sum over all vintages including the not-yet-existing one $q+1$ as we need to include the development of κ in (63).

where the last equality used (13). As (15) tells us $p_I B^j = B^q v_j$ and $p_c = p_I$ by (3), we can replace B^j by $B^j = B^q v_j / p_c$ and obtain

$$\begin{aligned} da &= \left(B^q \frac{\partial Y}{\partial K} \Sigma_{j=0}^{q+1} \frac{v_j}{p_Y} k_j - \delta a + w - i - c \right) dt + \left(\kappa \frac{i}{R} - \frac{B-1}{B} a \right) dq \\ &= (ra + w - i - c)\, dt + \left(\kappa \frac{i}{R} - sa \right) dq, \end{aligned}$$

where the interest rate r and s are defined as $r \equiv B^q \frac{\partial Y}{\partial K} - \delta$ and $s \equiv \frac{B-1}{B}$.

9.3 The cyclical components in section 3.1

9.3.1 The Bellman equation and first order conditions

The Bellman equations is (cf. e.g. Dixit and Pindyck, 1994, Sennewald, 2004, or Sennewald and Wälde, 2004)

$$\rho V(a,q) = \max \left\{ u(c) + V_a(a,q)\,[ra + w - i - c] + \lambda \left[V(\tilde{a}, q+1) - V(a,q) \right] \right\} \tag{64}$$

$$\text{where} \qquad \tilde{a} = (1-s)\, a + \kappa \frac{i}{R}. \tag{65}$$

The first order conditions for consumption and real investment i in R&D are

$$u'(c) = V_a(a,q), \tag{66}$$

$$V_a(a,q) = \lambda \frac{V(\tilde{a}, q+1)}{di} \Leftrightarrow V_a(a,q) = \lambda V_{\tilde{a}}(\tilde{a}, q+1)\, \kappa \frac{1}{R}. \tag{67}$$

9.3.2 The Keynes-Ramsey rule

(This section uses, combines and extends Wälde, 1999 and Wälde, 2005). The marginal value of a unit of wealth $V_a(a,q)$ is a function of both assets a and of the technological level q. The differential of the marginal value reads

$$dV_a(a,q) = V_{aa}(a,q)\,[ra + w - i - c]\, dt + \left[V_{\tilde{a}}(\tilde{a}, q+1) - V_a(a,q) \right] dq \tag{68}$$

where $\tilde{a}$ is as in (65). The partial derivative of the maximized Bellman equation using the envelope theorem reads

$$\begin{aligned} \rho V_a(a,q) &= V_{aa}(a,q)\,[ra + w - i - c] + V_a(a,q)\, r + \lambda \left[V_a(\tilde{a}, q+1) - V_a(a,q) \right] \\ &= V_{aa}(a,q)\,[ra + w - i - c] + V_a(a,q)\, r + \lambda \left[(1-s)\, V_{\tilde{a}}(\tilde{a}, q+1) - V_a(a,q) \right] \end{aligned}$$

where the last equality used (65). With $V_a \equiv V_a(a,q)$, $V_{aa} \equiv V_{aa}(a,q)$ and $V_{\tilde{a}} \equiv V_{\tilde{a}}(\tilde{a}, q+1)$ and rearranging this reads $[\rho - r + \lambda]\, V_a - \lambda (1-s)\, V_{\tilde{a}} = V_{aa}\,[ra + w - i - c]$. Replacing $V_{aa}\,[ra + w - i - c]$ in (68) by this expression gives

$$dV_a = \left[(\rho - r + \lambda)\, V_a - (1-s)\, \lambda V_{\tilde{a}} \right] dt + \left[V_{\tilde{a}} - V_a \right] dq \tag{69}$$

and with V_a being replaced by marginal utility $u'(c)$ from (66)

$$\begin{aligned}du'(c) &= [(\rho - r + \lambda) u'(c) - (1-s)\lambda u'(\tilde{c})]\,dt + [u'(\tilde{c}) - u'(c)]\,dq \\ &= [(\rho - r) u'(c) + \lambda [u'(c) - (1-s) u'(\tilde{c})]]\,dt + [u'(\tilde{c}) - u'(c)]\,dq.\end{aligned}$$

Now let $f(.)$ be the inverse function for $u'(c)$, i.e. $f(u'(c)) = c$, and apply Ito's Lemma to $f(u'(c))$. As $f(u'(c)) = c$, $f'(.) = \frac{df(u'(c))}{du'(c)} = \frac{dc}{du'(c)} = \frac{1}{u''(c)}$ and $f(u'(\tilde{c})) = \tilde{c}$, this yields

$$\begin{aligned}df(u'(c)) &= \frac{1}{u''(c)}\{u'(c)[\rho - r] + \lambda[u'(c) - (1-s)u'(\tilde{c})]\}\,dt + [\tilde{c} - c]\,dq \Leftrightarrow \\ dc &= -\frac{u'(c)}{u''(c)}\left\{r - \rho - \lambda\left[1 - (1-s)\frac{u'(\tilde{c})}{u'(c)}\right]\right\}dt + \{\tilde{c} - c\}\,dq. \qquad (70)\end{aligned}$$

9.3.3 Cyclical components (24) to (27)

This section derives the reduced form in the cyclical components $\hat{K}$ and $\hat{C}$, where $K = \hat{K}A^{q/\alpha}$ and $C = \hat{C}A^q$ as in (23). Assuming a representative agent, we aggregate by replacing individual consumption c by aggregate consumption C. We start with (24).

The differential of $\hat{C}$ is, given the Keynes-Ramsey rule in (70) and using Ito's Lemma,

$$\begin{aligned}d\hat{C} = d\left(CA^{-q}\right) &= A^{-q}\left[-\frac{u'(C)}{u''(C)}\left\{r - \rho - \lambda\left[1 - (1-s)\frac{u'\left(\tilde{\hat{C}}A^{q+1}\right)}{u'\left(\hat{C}A^q\right)}\right]\right\}\right]dt \\ &+ \left\{\tilde{C}A^{-(q+1)} - CA^{-q}\right\}dq \\ &= -\frac{u'\left(\hat{C}\right)}{u''\left(\hat{C}\right)}\left\{r - \rho - \lambda\left[1 - (1-s)\frac{u'\left(A\tilde{\hat{C}}\right)}{u'\left(\hat{C}\right)}\right]\right\}dt + \left\{\tilde{\hat{C}} - \hat{C}\right\}dq\end{aligned}$$

where for the last step we used properties of the CES instantaneous utility function (19). This equation is equivalent to (24) in the text.

We now derive (25). With (66) and rearranging, we get for the first order condition for R&D investment (67)

$$u'(C) = \lambda u'\left(\tilde{C}\right)\kappa\frac{1}{R}. \qquad (71)$$

The ratio of marginal utilities is in terms of cyclical components with (23)

$$\frac{u'\left(\tilde{C}\right)}{u'(C)} = \left(\frac{C}{\tilde{C}}\right)^{\sigma} = \left(\frac{\hat{C}A^q}{\tilde{\hat{C}}A^{q+1}}\right)^{\sigma} = \left(\frac{\hat{C}}{A\tilde{\hat{C}}}\right)^{\sigma} = \frac{u'\left(A\tilde{\hat{C}}\right)}{u'\left(\hat{C}\right)}, \qquad (72)$$

where the first and last equality sign uses the definition of the instantaneous utility function in (19). Inserting κ from (7) and R from (4) with D from (5), we get for the RHS of (71)

$$\frac{\lambda\kappa}{R} = \frac{\lambda\kappa}{\lambda^{1/(1-\gamma)}D} = \lambda^{-\gamma/(1-\gamma)}\frac{\kappa}{D} = \lambda^{-\gamma/(1-\gamma)}\frac{\kappa_0}{D_0} \qquad (73)$$

By inserting (72) and (73) into (71) we get (25) in the text.

Third we derive (26). As from (4) and (5)

$$R = \lambda^{1/(1-\gamma)} D = \lambda^{1/(1-\gamma)} D_0 K^c, \tag{74}$$

and by (17) and (23)

$$K^c = B^{-q} K = B^{-q} \hat{K} A^{q/\alpha} = A^q \hat{K} \tag{75}$$

productivity adjusted resources allocated to R&D are independent of q as well,

$$\hat{R} \equiv A^{-q} R = \lambda^{1/(1-\gamma)} D_0 \hat{K}. \tag{76}$$

Finally, to get (27), combine the equation (14) describing the evolution of the capital index together with the goods market clearing condition (2), yielding

$$dK = \left(B^q \left[Y - R - C\right] - \delta K\right) dt + B^{q+1} \kappa dq. \tag{77}$$

Again, with Ito's Lemma,

$$\begin{aligned} d\hat{K} = d\left(K A^{-q/\alpha}\right) &= \left\{A^{-q/\alpha} B^q \left[Y - R - C\right] - \delta \hat{K}\right\} dt \\ &+ \left\{\left(K + B^{q+1}\kappa\right) A^{-(q+1)/\alpha} - K A^{-q/\alpha}\right\} dq \\ &= \left\{\hat{Y} - A^{-q} R - \hat{C} - \delta \hat{K}\right\} dt + \left\{A^{-1/\alpha} + A^{-1}\kappa_0 - 1\right\} \hat{K} dq \end{aligned}$$

where we used

$$A^{q/\alpha} B^{-q} = A^{q/\alpha} A^{-\frac{1-\alpha}{\alpha} q} = A^q \tag{78}$$

which implies $A^{-q/\alpha} B^q C = \hat{C}$ and $A^{-q/\alpha} B^q Y = \hat{K}^\alpha L^{1-\alpha} \equiv \hat{Y}$ for the deterministic part. For the stochastic part, we used (7), (17) and (23) and got

$$\begin{aligned} \left(K + B^{q+1}\kappa\right) A^{-(q+1)/\alpha} - K A^{-q/\alpha} &= \left(K + B^{q+1}\kappa_0 B^{-q} K\right) A^{-q/\alpha} A^{-1/\alpha} - K A^{-q/\alpha} \\ &= \left(A^{-1/\alpha} + A^{-1}\kappa_0 - 1\right) \hat{K}. \end{aligned}$$

9.4 Proof of theorem 1 (linear policy rule)

The proof proceeds as follows. We first assume that a solution exists where consumption is a constant share Ψ of capital, $\hat{C} = \Psi \hat{K}$. Then we show under which parameter restrictions this is consistent with equilibrium and optimality conditions.[24]

(i) From (27), the jump of capital is constant,

$$\tilde{\hat{K}} / \hat{K} = A^{-1}\kappa_0 + A^{-1/\alpha}. \tag{79}$$

Now assume $\hat{C} = \Psi \hat{K}$. Then, by (79)

$$\tilde{\hat{C}} / \hat{C} = \tilde{\hat{K}} / \hat{K} = A^{-1}\kappa_0 + A^{-1/\alpha} \equiv A^{-1}\xi, \tag{80}$$

[24] One could alternatively guess a value function and show that it satisfies the Bellman equation and first order conditions. This alternative approach would provide a proof of a verification theorem and leads of course to identical results.

i.e. consumption jumps in the same way as capital. This is (33) and the definition of ξ with (12) is (30) in the theorem.

As this jump is constant, the arrival rate by the investment first order condition (25) is constant as well. Inserting ξ from (80) into (25) gives $\xi^{\sigma} = \lambda^{-\gamma/(1-\gamma)}\kappa_0 D_0^{-1} \Leftrightarrow \lambda = \left(\xi^{-\sigma}\kappa_0 D_0^{-1}\right)^{(1-\gamma)/\gamma}$. This is (29) in the theorem.

(ii) Our system (24) to (27) reduces by inserting (26) and (80) to a set of two equations determining $\hat{C}$ and $\hat{K}$ as a function of $\hat{C}$ and $\hat{K}$ and constant parameters $(\lambda, \xi, \kappa_0, D_0)$. Using the abbreviations

$$\hat{\rho} = \rho + \lambda\left[1 - (1-s)\,\xi^{-\sigma}\right], \quad \hat{\delta} = \delta + \lambda^{1/(1-\gamma)} D_0, \tag{81}$$

this reads

$$-\frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)}d\hat{C} = \{r - \hat{\rho}\}\,dt - \frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)}\left\{A^{-1}\xi - 1\right\}\hat{C}dq,$$

$$d\hat{K} = \left\{\hat{Y} - \hat{C} - \hat{\delta}\hat{K}\right\}dt + \left\{A^{-1/\alpha} + A^{-1}\kappa_0 - 1\right\}\hat{K}dq.$$

(iii) We now show that with $\alpha = \sigma$, consumption is a linear function of the capital index. Assume, this is the case. Then,

$$\frac{\dot{\hat{C}}}{\hat{C}} = \frac{\dot{\hat{K}}}{\hat{K}} \Leftrightarrow \frac{r - \hat{\rho}}{\sigma} = \frac{\hat{Y} - \hat{C} - \hat{\delta}\hat{K}}{\hat{K}} \Leftrightarrow$$

$$\left(\frac{\partial \hat{Y}}{\partial \hat{K}} - \delta\right)\hat{K} - \hat{\rho}\hat{K} = \sigma\hat{Y} - \sigma\Psi\hat{K} - \sigma\left(\delta + \lambda^{1/(1-\gamma)}D_0\right)\hat{K} \Leftrightarrow$$

$$(\alpha - \sigma)\,\hat{Y} = \hat{K}\left[\hat{\rho} - \sigma\left(\Psi + \lambda^{1/(1-\gamma)}D_0\right) + (1-\sigma)\,\delta\right].$$

A sufficient condition is therefore $\sigma = \alpha$ *and* $\Psi = \frac{\hat{\rho}+(1-\sigma)\delta}{\sigma} - \lambda^{1/(1-\gamma)}D_0$. Reinserting (81), this proves (32) and thereby (31).

9.5 Results related to figure 2

As the observed capital stock (17) is by (75) proportional to K, K^c behaves quantitatively in an identical way as the capital index K as long as no jump occurs. Computing the jump of the observed capital stock by applying (75) to (33), however, shows with (17)

$$A^{-1}\xi = \frac{\tilde{K}^c A^{-(q+1)}}{K^c A^{-q}} \Leftrightarrow \xi = \frac{\tilde{K}^c}{K^c} = B^{-1}\frac{\tilde{K}}{K}. \tag{82}$$

Resources allocated to R&D are given by (74) and (75) by

$$R = \lambda^{1/(1-\gamma)} D_0 A^{-q}\hat{K}. \tag{83}$$

Over the cycle, they increase as capital does. The behaviour after a jump is not obvious as A^q increases but $\hat{K}$ decreases. Compute with (33)

$$\frac{\tilde{R}}{R}=\frac{\tilde{\hat{K}}A^{q+1}}{\hat{K}A^q}=\xi=\frac{\tilde{K}^c}{K^c} \tag{84}$$

As this is the same expression as for the observed capital stock (82), the same arguments apply here.

Aggregate consumption changes according to (33). With (23), we have

$$\frac{\tilde{C}}{C}=\xi=\frac{\tilde{K}^c}{K^c}. \tag{85}$$

Investment is given by (2) as $I=Y-R-C$. The jump in investment can be deduced with (10), (82), (84) and (85) from

$$\begin{aligned}\frac{\tilde{I}}{I}&=\frac{B^{q+1}\left[(B\xi)^{\alpha}Y-\xi\left[R+C\right]\right]-\delta B\xi K}{B^q\left[Y-R-C\right]-\delta K}=\frac{B\left[(B\xi)^{-(1-\alpha)}Y-B^{-1}\left[R+C\right]\right]-B^{-q}\delta K}{Y-R-C-B^{-q}\delta K}B\xi\\&=\frac{B\left(B\xi\right)^{-(1-\alpha)}Y-R-C-B^{-q}\delta K}{Y-R-C-B^{-q}\delta K}B\xi>1.\end{aligned}$$

The inequality follows from the following reasoning. As $B\xi>1$ by (30) and $B\left(B\xi\right)^{-(1-\alpha)}=B^{\alpha}\xi^{-(1-\alpha)}=\left(A^{-1}\xi\right)^{-(1-\alpha)}>1$ as well by (35), $\tilde{I}/I>1$.

9.6 The expected growth rate (40)

The growth rate between t and T is

$$\begin{aligned}g_{T,t}&\equiv\ln Y\left(T\right)-\ln Y\left(t\right)\\&=q\left(T\right)\ln A+\alpha\ln\hat{K}\left(T\right)+\left(1-\alpha\right)\ln L-\left[q\left(t\right)\ln A+\alpha\ln\hat{K}\left(t\right)+\left(1-\alpha\right)\ln L\right]\\&=\left(q\left(T\right)-q\left(t\right)\right)\ln A+\alpha\left[\ln\hat{K}\left(T\right)-\ln\hat{K}\left(t\right)\right].\end{aligned}$$

Using $E\left[q\left(t\right)-\lambda t\right]=0$, i.e. the martingale property of $q\left(t\right)-\lambda t$, the expected growth rate for a period of length $T-t$ is then $E_t g_{T,t}=\ln A\lambda\left[T-t\right]+\alpha\left[E_t\ln\hat{K}\left(T\right)-\ln\hat{K}\left(t\right)\right]$. Computing the expected growth rate per unit of time gives $Eg_t\equiv\frac{E_t g_{T,t}}{T-t}=\lambda\ln A+\alpha\frac{E_t\ln\hat{K}(T)-\ln\hat{K}(t)}{T-t}$.

9.7 The derivative of the arrival rate (41)

In order to understand the sign of the derivative of the arrival rate (41) with respect to κ_0 under $\gamma>0$, it is enough to understand the sign of the following derivative,

$$\begin{aligned}\frac{d}{d\kappa_0}\frac{\kappa_0}{\left[\kappa_0+B^{-1}\right]^{\sigma}}=\frac{\left[\kappa_0+B^{-1}\right]^{\sigma}-\sigma\left[\kappa_0+B^{-1}\right]^{\sigma-1}\kappa_0}{\left[\kappa_0+B^{-1}\right]^{2\sigma}}>0\Leftrightarrow\\\kappa_0+B^{-1}>\sigma\kappa_0\Leftrightarrow\left(1-\sigma\right)\kappa_0+B^{-1}>0.\end{aligned}$$

This is positive e.g. for our assumption $\sigma=\alpha<1$.

9.8 An optimal constant saving rate

This proves theorem 2. The structure of the proof is identical to the proof of theorem 1 in 9.4. We again assume that a solution exists where, now, consumption is a constant share Ψ of output, $\hat{C} = \Psi \hat{K}^{\alpha} L^{1-\alpha}$. Then we show under which parameter restrictions this is consistent with equilibrium and optimality conditions.

(i) As before, the jump of capital is given by (79). Assuming $\hat{C} = \Psi \hat{K}^{\alpha} L^{1-\alpha}$,

$$\frac{\tilde{\hat{C}}}{\hat{C}} = \left(\frac{\tilde{\hat{K}}}{\hat{K}}\right)^{\alpha} = \left(A^{-1}\kappa_0 + A^{-1/\alpha}\right)^{\alpha} = \left(A^{-1}\xi\right)^{\alpha}, \tag{86}$$

where ξ is defined as in (30).

The arrival rate by the investment first order condition (25) is again constant. Inserting (86) into (25) gives

$$\left(A\left(A^{-1}\xi\right)^{\alpha}\right)^{\sigma} = \lambda^{-\gamma/(1-\gamma)}\kappa_0/D_0 \Leftrightarrow \lambda = \left(\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\kappa_0/D_0\right)^{(1-\gamma)/\gamma}. \tag{87}$$

(ii) Our system (24) to (27) reduces by inserting (26) and (86) to a set of two equations determining $\hat{C}$ and $\hat{K}$ as a function of $\hat{C}$ and $\hat{K}$ and constant parameters $(\lambda, \xi, \kappa_0, D_0)$. Using the abbreviations

$$\bar{\rho} = \rho + \lambda\left[1 - (1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\right], \quad \bar{\delta} = \delta + \lambda^{1/(1-\gamma)}D_0, \tag{88}$$

we get

$$\begin{aligned} -\frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)}d\hat{C} &= \{r - \bar{\rho}\}\,dt - \frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)}\left\{\left(A^{-1}\xi\right)^{\alpha} - 1\right\}\hat{C}dq, \\ d\hat{K} &= \left\{\hat{Y} - \hat{C} - \bar{\delta}\hat{K}\right\}dt + \left\{A^{-1/\alpha} + A^{-1}\kappa_0 - 1\right\}\hat{K}dq. \end{aligned}$$

(iii) Given our assumption $\hat{C} = \Psi \hat{K}^{\alpha} L^{1-\alpha}$,

$$\begin{aligned} \frac{\dot{\hat{C}}}{\hat{C}} &= \alpha\frac{\dot{\hat{K}}}{\hat{K}} \Leftrightarrow \frac{r-\bar{\rho}}{\sigma} = \alpha\frac{\hat{Y} - \Psi\hat{K}^{\alpha}L^{1-\alpha} - \bar{\delta}\hat{K}}{\hat{K}} \Leftrightarrow \\ \left(\frac{\partial\hat{Y}}{\partial\hat{K}} - \delta\right)\hat{K} - \bar{\rho}\hat{K} &= \sigma\alpha\hat{Y} - \sigma\alpha\Psi\hat{K}^{\alpha}L^{1-\alpha} - \sigma\alpha\left(\delta + \lambda^{1/(1-\gamma)}D_0\right)\hat{K} \Leftrightarrow \\ (\alpha - \alpha\sigma + \alpha\sigma\Psi)\,\hat{Y} &= \left(\delta + \bar{\rho} - \sigma\alpha\left(\delta + \lambda^{1/(1-\gamma)}D_0\right)\right)\hat{K} \\ &= \left((1-\sigma\alpha)\,\delta + \bar{\rho} - \sigma\lambda^{1/(1-\gamma)}D_0\right)\hat{K}. \end{aligned}$$

This holds for $\Psi = \frac{\sigma-1}{\sigma}$ *and* $(1-\sigma\alpha)\,\delta + \bar{\rho} = \sigma\lambda^{1/(1-\gamma)}D_0$ as this makes both sides equal to zero. While the first condition requires a saving rate of $s = 1 - \Psi = 1/\sigma$, the second condition requires with (88)

$$(1-\alpha\sigma)\,\delta + \rho + \lambda\left[1 - (1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\right] = \sigma\lambda^{1/(1-\gamma)}D_0 \tag{89}$$

which is (46) in the text. This completes the proof of the theorem.

In order to better understand (89), we now rearrange $\lambda\left[1-(1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\right] = \sigma\lambda^{1/(1-\gamma)}D_0$ and then insert the expression for the arrival rate (87),

$$\begin{aligned}
\lambda\left[1-(1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\right] &= \sigma\lambda^{1/(1-\gamma)}D_0 \Leftrightarrow \\
\lambda\left[1-(1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\right]-\sigma\lambda^{1/(1-\gamma)}D_0 &= 0 \Leftrightarrow \\
\lambda\left[1-(1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}-\sigma\lambda^{\gamma/(1-\gamma)}D_0\right] &= 0 \Leftrightarrow \\
\lambda\left[1-(1-s)\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}-\sigma\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\kappa_0\right] &= 0 \Leftrightarrow \\
\lambda\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\left[\left(A^{1-\alpha}\xi^{\alpha}\right)^{\sigma}-(1-s)-\sigma\kappa_0\right] &= 0 \Leftrightarrow \\
\lambda\left(A^{1-\alpha}\xi^{\alpha}\right)^{-\sigma}\left[A^{(1-\alpha)\sigma}\left(\kappa_0+B^{-1}\right)^{\alpha\sigma}-B^{-1}-\sigma\kappa_0\right] &= 0
\end{aligned}$$

As κ_0 is close to zero, approximating the expression in squared brackets for $\kappa_0 = 0$ gives $\left[A^{(1-\alpha)\sigma}B^{-\alpha\sigma}-B^{-1}\right] = 1-B^{-1}$. As this is close to zero, σ implied by (89) is similar to its value in deterministic models as discussed in the text.

The jump in consumption, referred to in footnote 20, follows from (86) with (23) and (30)

$$\frac{\tilde{C}}{C} = \frac{\tilde{\hat{C}}A^{q+1}}{\hat{C}A^{q}} = A\left(A^{-1}\xi\right)^{\alpha} = \left(B\left[\kappa_0+B^{-1}\right]\right)^{\alpha} = \left(B\kappa_0+1\right)^{\alpha}.$$

9.9 An alternative difficulty function

9.9.1 The new reduced form

When the difficulty function is given by (6), the reduced form system (24) to (27) changes. The Keynes-Ramsey rule (24) remains unchanged. As (73) now reads $\frac{\lambda\kappa}{R} = \frac{\lambda\kappa}{\lambda^{1/(1-\gamma)}D} = \lambda^{-\gamma/(1-\gamma)}\frac{\kappa}{D} = \lambda^{-\gamma/(1-\gamma)}\frac{\kappa_0\hat{K}}{D_0}$, (25) becomes

$$u'\left(\hat{C}\right) = u'\left(A\tilde{\hat{C}}\right)\lambda^{-\gamma/(1-\gamma)}\frac{\kappa_0\hat{K}}{D_0}. \tag{90}$$

The third equation (26) changes as from (4) and (6)

$$R = \lambda^{1/(1-\gamma)}D = \lambda^{1/(1-\gamma)}D_0A^q \tag{91}$$

and therefore

$$\hat{R} \equiv A^{-q}R = \lambda^{1/(1-\gamma)}D_0. \tag{92}$$

while (27) remains unchanged.

9.9.2 The proof of theorem 3

(i) Assume a solution where $\hat{C} = \Psi \hat{K}$ exists.[25] Then $\tilde{\hat{C}}/\hat{C} = \tilde{\hat{K}}/\hat{K} = A^{-1}\kappa_0 + A^{-1/\alpha} \equiv A^{-1}\xi$ as in (80) and with (90)

$$\xi^{\sigma} = \lambda^{-\gamma/(1-\gamma)} \frac{\kappa_0 \hat{K}}{D_0} \Leftrightarrow \lambda = \left(\xi^{-\sigma} \frac{\kappa_0 \hat{K}}{D_0} \right)^{(1-\gamma)/\gamma} . \tag{93}$$

(ii) With (92), the resource constraint (27) reads

$$d\hat{K} = \left\{ \hat{Y} - \lambda^{1/(1-\gamma)} D_0 - \hat{C} - \delta \hat{K} \right\} dt + \left\{ A^{-1/\alpha} + A^{-1}\kappa_0 - 1 \right\} \hat{K} dq.$$

Inserting (90) into the Keynes-Ramsey rule (24) gives with (93) for the last equality

$$\begin{aligned}
-\frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)} d\hat{C} &= \left\{ \alpha \left(L/\hat{K} \right)^{1-\alpha} - \delta - \rho - \lambda \left[1 - (1-s)\, \lambda^{\gamma/(1-\gamma)} \frac{D_0}{\kappa_0 \hat{K}} \right] \right\} dt \\
&\quad - \frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)} \left\{ \tilde{\hat{C}} - \hat{C} \right\} dq \\
&= \left\{ \alpha \left(L/\hat{K} \right)^{1-\alpha} - \delta - \rho - \lambda \left[1 - (1-s)\, \xi^{-\sigma} \right] \right\} dt - \frac{u''\left(\hat{C}\right)}{u'\left(\hat{C}\right)} \left\{ \tilde{\hat{C}} - \hat{C} \right\} dq.
\end{aligned}$$

(iii) Hence, with $\hat{C} = \Psi \hat{K}$,

$$\begin{aligned}
\frac{\dot{\hat{C}}}{\hat{C}} = \frac{\dot{\hat{K}}}{\hat{K}} \Leftrightarrow \frac{\alpha \left(L/\hat{K} \right)^{1-\alpha} - \delta - \rho - \lambda \left[1 - (1-s)\, \xi^{-\sigma} \right]}{\sigma} \\
= \frac{\hat{Y} - \lambda^{1/(1-\gamma)} D_0 - (\Psi + \delta) \hat{K}}{\hat{K}} \Leftrightarrow \\
\alpha \hat{Y} - (\delta + \rho) \hat{K} - \lambda \left[1 - (1-s)\, \xi^{-\sigma} \right] \hat{K} = \sigma \hat{Y} - \sigma \lambda^{1/(1-\gamma)} D_0 - \sigma (\Psi + \delta) \hat{K} \Leftrightarrow \\
(\alpha - \sigma) \hat{Y} - \lambda \left[1 - (1-s)\, \xi^{-\sigma} \right] \hat{K} + \sigma \lambda^{1/(1-\gamma)} D_0 = (\delta + \rho - \sigma (\Psi + \delta)) \hat{K}.
\end{aligned}$$

[25] We present no solution for $\hat{C} = \bar{\Psi} \hat{K}^{\alpha} L^{1-\alpha}$. Such a solution can not easily be found for the present setup but is straightforward if an additional parameter, say a R&D subsidy financed through lump-sum tax, is introduced. Results concerning the arrival rate and cyclical behaviour of R&D expenditure are then qualitatively identical.

This holds for $\alpha = \sigma$, $\Psi = \frac{(1-\sigma)\delta+\rho}{\sigma}$ and

$$\begin{aligned}
\lambda \left[1-(1-s)\,\xi^{-\sigma}\right]\hat{K} &= \sigma\lambda^{1/(1-\gamma)}D_0 \Leftrightarrow \\
\left(\xi^{-\sigma}\frac{\kappa_0\hat{K}}{D_0}\right)^{(1-\gamma)/\gamma}\left[1-(1-s)\,\xi^{-\sigma}\right]\hat{K} &= \sigma\left(\xi^{-\sigma}\frac{\kappa_0\hat{K}}{D_0}\right)^{1/\gamma}D_0 \Leftrightarrow \\
\left(\xi^{-\sigma}\kappa_0\right)^{(1-\gamma)/\gamma}\left[1-(1-s)\,\xi^{-\sigma}\right] &= \sigma\left(\xi^{-\sigma}\kappa_0\right)^{1/\gamma} \Leftrightarrow \\
1-(1-s)\,\xi^{-\sigma} &= \sigma\xi^{-\sigma}\kappa_0 \Leftrightarrow 1 = (1-s+\sigma\kappa_0)\,\xi^{-\sigma} \Leftrightarrow \\
\left(B^{-1}+\kappa_0\right)^{\sigma} &= B^{-1}+\sigma\kappa_0
\end{aligned}$$

where we used $1-s = B^{-1}$ from (22) and the definition of ξ in (30). For reasonable parameter values, this holds for a certain κ_0 lying in the required interval [0,1]. This completes the proof of 3.

Inserting (93) in (91) gives $R = \left(\xi^{-\sigma}\frac{\kappa_0\hat{K}}{D_0}\right)^{1/\gamma} D_0A^q$ which implies a cyclical component of capital of $\hat{R} = \left(\xi^{-\sigma}\frac{\kappa_0\hat{K}}{D_0}\right)^{1/\gamma} D_0$ as in the text.

9.10 A multi-sector version

Labour's marginal productivity in sector j and k are

$$\begin{aligned}
\frac{\partial Y}{\partial L_j} &= \left[\Pi_{i=1,i\neq j}^{N}\left(\Gamma_i K_i^{\alpha_i}(A^{q_i}L_i)^{1-\alpha_i}\right)^{\gamma_i}\right]\left(\Gamma_j K_j^{\alpha_j}A^{q_j[1-\alpha_j]}\right)^{\gamma_j}(1-\alpha_j)\,\gamma_j L_j^{(1-\alpha_j)\gamma_j-1}, \\
\frac{\partial Y}{\partial L_k} &= \left[\Pi_{i=1,i\neq k}^{N}\left(\Gamma_i K_i^{\alpha_i}(A^{q_i}L_i)^{1-\alpha_i}\right)^{\gamma_i}\right]\left(\Gamma_k K_k^{\alpha_k}A^{q_k[1-\alpha_k]}\right)^{\gamma_k}(1-\alpha_k)\,\gamma_k L_k^{(1-\alpha_k)\gamma_k-1}.
\end{aligned}$$

These are the same when

$$\begin{gathered}
\left(\Gamma_k K_k^{\alpha_k}(A^{q_k}L_k)^{1-\alpha_k}\right)^{\gamma_k}\left(\Gamma_j K_j^{\alpha_j}A^{q_j[1-\alpha_j]}\right)^{\gamma_j}(1-\alpha_j)\,\gamma_j L_j^{(1-\alpha_j)\gamma_j-1} \\
= \left(\Gamma_j K_j^{\alpha_j}(A^{q_j}L_j)^{1-\alpha_j}\right)^{\gamma_j}\left(\Gamma_k K_k^{\alpha_k}A^{q_k[1-\alpha_k]}\right)^{\gamma_k}(1-\alpha_k)\,\gamma_k L_k^{(1-\alpha_k)\gamma_k-1} \Leftrightarrow \\
L_k(1-\alpha_j)\,\gamma_j = L_j(1-\alpha_k)\,\gamma_k
\end{gathered}$$

Labour market clearing requires $L_1 + L_2 + ... + L_N = L$ which implies upon inserting

$$\begin{gathered}
L_1 + \frac{(1-\alpha_2)\,\gamma_2}{(1-\alpha_1)\,\gamma_1}L_1 + ... + \frac{(1-\alpha_N)\,\gamma_N}{(1-\alpha_1)\,\gamma_1}L_1 = L \Leftrightarrow \\
((1-\alpha_1)\,\gamma_1 + (1-\alpha_2)\,\gamma_2 + ... + (1-\alpha_N)\,\gamma_N)\,\frac{L_1}{(1-\alpha_1)\,\gamma_1} = L <\equiv> \\
L_1 = \frac{(1-\alpha_1)\,\gamma_1}{\Gamma}L.
\end{gathered}$$

Doing the same for capital gives $K_i = \frac{\alpha_i\gamma_i}{\Delta}K$, given an appropriate definition for Δ. By putting all constants, including Γ_i, into Y_0, the aggregate technology (51) becomes (53) in the text,

$$Y = Y_0\Pi_{i=1}^{N}\left(K^{\alpha_i}L^{1-\alpha_i}\right)^{\gamma_i}\Pi_{i=1}^{N}A^{q_i(1-\alpha_i)\gamma_i} = Y_0A^{\Sigma_{i=1}^{N}q_i(1-\alpha_i)\gamma_i}K^{\Sigma_{i=1}^{N}\alpha_i\gamma_i}L^{\Sigma_{i=1}^{N}(1-\alpha_i)\gamma_i}.$$

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