A Appendix

Proofs for Stress and Coping - An Economic Approach
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A.1 Solution of the maximization problem

A.1.1 The Bellman equation

We start from the general specification of a Bellman equation in continuous time under uncertainty (see Wälde, 1999 or Sennewald, 2007). Given the objective function (11), the state variable $W(t)$ and the control $m(t)$, it reads

$$\rho V(W(t)) = \max_{m(t)} \left\{ u(W(t)) - v(m(t)) + \frac{1}{dt} E_t dV(W(t)) \right\}. \tag{A.1}$$

The differential of the value function reads, given the law of motion in (12) for $W(t)$ (see e.g. Wälde, 2012, ch. 10.2.3),

$$\begin{aligned} dV(W(t)) = {} & V'(W(t)) \left\{ \phi \frac{p}{b} W(t) - \delta_0 W(t) - \delta_1 m(t) \right\} dt \\ & + \{V(W(t) - \chi g(t)) - V(W(t))\} dq_g(t) \\ & + \{V(W(t) - \Delta(t)) - V(W(t))\} dq_\Delta(t). \end{aligned}$$

Forming expectations E_t yields

$$\begin{aligned} dV(W(t)) = {} & V'(W(t)) \left\{ \phi \frac{p}{b} W(t) - \delta_0 W(t) - \delta_1 m(t) \right\} dt \\ & + E^h \{V(W(t) - \chi g(t)) - V(W(t))\} \lambda^g dt \\ & + E^\Delta \{V(W(t) - \Delta(t)) - V(W(t))\} \lambda^\Delta (W(t)) dt, \end{aligned} \tag{A.2}$$

where we assume that the jump size of $g(t)$ in (1) and the frequency of jumps of $q_g(t)$ are independent, as is the stress effect $\Delta(t)$ of an outburst and the corresponding Poisson process $q_\Delta(t)$. All sources of uncertainty are taken into account when the expectations operator E_t is applied. As the expected numbers of jump of $q_g(t)$ over a small time interval dt is given by $\lambda^g dt$, only the expectations with respect to $g(t)$ need to be formed. They are denoted by E^h as this expectation refers to the random size of $h(t)$ in (1). The same reasoning holds for random outburst effects $\Delta(t)$ where expectations are denoted by E^Δ.

Using the definition of Φ in (13), dividing by dt and replacing this expression in the above general specification (A.1) yields

$$\rho V(W(t)) = \max_{m(t)} \left\{ \begin{array}{c} u(W(t)) - v_0 m(t)^{1+\zeta} + V'(W(t)) [\Phi W(t) - \delta_1 m(t)] \\ +\lambda^g \left[E^h V(W(t) - \chi g(t)) - V(W(t)) \right] \\ +\lambda^\Delta (W(t)) \left[E^h V(W(t) - \Delta(t)) - V(W(t)) \right] \end{array} \right\}. \tag{A.3}$$

The utility function $u(W(t))$ is given by the expression from (10).

The first-order condition for $m(t)$ in (A.3) requires

$$(1+\zeta) v_0 m(t)^\zeta = -V'(W(t)) \delta_1. \tag{A.4}$$

Assuming an interior solution, marginal costs from coping on the left-hand side must equal marginal gains on the right. As stress $W(t)$ is "a bad" and not a good, $V'(W(t))$ is negative and the right-hand side of this first-order condition indeed reflects a positive marginal gain.

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A.1.2 Solving by guess and verify

We start with a guess $J(W)$ for the value function $V(W)$ which reads $J(W) = \Lambda_0 - \Lambda_1 W$. The guess implies $J'(W) = -\Lambda_1$ and gives us two free parameters, Λ_0 and Λ_1. We need to verify that this guess satisfies the first-order condition and the Bellman equation.

The first-order condition (A.4) is satisfied if

$$(1+\zeta) v_0 m^{\zeta} = \Lambda_1 \delta_1. \tag{A.5}$$

Concerning the Bellman equation and given the guess, the jump terms read

$$\begin{aligned} E^h V(W-\chi g) - V(W) &= E^h (\Lambda_0 - \Lambda_1 [W-\chi g]) - (\Lambda_0 - \Lambda_1 W) \\ &= \Lambda_0 - \Lambda_1 \left[W - \chi E^h g\right] - \Lambda_0 + \Lambda_1 W \\ &= \Lambda_1 \chi E^h g \end{aligned}$$

and

$$\begin{aligned} E^{\Delta} V(W-\Delta) - V(W(t)) &= E^{\Delta} (\Lambda_0 - \Lambda_1 [W-\Delta]) - (\Lambda_0 - \Lambda_1 W) \\ &= \Lambda_0 - \Lambda_1 \left[W - \mu^{\Delta}\right] - \Lambda_0 + \Lambda_1 W \\ &= \Lambda_1 \mu^{\Delta}. \end{aligned}$$

The Bellman equation (A.3) as a whole then reads

$$\begin{aligned} &\rho [\Lambda_0 - \Lambda_1 W] \\ &= \eta w [M - \kappa W] e - \alpha W - v_0 m^{1+\zeta} - \Lambda_1 [\Phi W - \delta_1 m] + \lambda^g \Lambda_1 \chi E^h g + \lambda^{\Delta}(W) \Lambda_1 \mu^{\Delta} \\ &= \eta w M e - (\eta w \kappa e + \alpha) W - v_0 m^{1+\zeta} - \Lambda_1 \left[\Phi W - \delta_1 m - \lambda^g \chi E^h g - \lambda^{\Delta}(W) \mu^{\Delta}\right] \end{aligned} \tag{A.6}$$

where the utility function from (10) for $W < W^s$ was employed.

Note that for $W \geq W^s$, the Bellman equation reads

$$\rho [\Lambda_0 - \Lambda_1 W] = -\alpha W - v_0 m^{1+\zeta} - \Lambda_1 \left[\Phi^s W - \delta_1 m - \lambda^g \chi E^h g - \lambda^{\Delta}(W) \mu^{\Delta}\right]$$

as $\eta w M - (\eta w \kappa + \alpha) W$ turns into $-\alpha W$ from (10). Only the direct negative effect of stress matters. The attention effect is no longer there as the individual is off the job. Mechanically speaking, the Bellman equation for $W > W^s$ is a special case of the Bellman equation for $W \leq W^s$ where $w = 0$ and $\Phi = \Phi^s$.

Now collect constant terms and terms proportional to W in (A.6),

$$\begin{aligned} \rho\Lambda_0 - \rho\Lambda_1 W = \eta w M e - (\eta w \kappa e + \alpha) W - v_0 m^{1+\zeta} - \Lambda_1 \Phi W - \Lambda_1 \left[-\delta_1 m - \lambda^g \chi E^h g - \lambda^{\Delta}(W) \mu^{\Delta}\right] \Leftrightarrow \\ \rho\Lambda_0 - \eta w M e + v_0 m^{1+\zeta} - \Lambda_1 \left[\delta_1 m + \lambda^g \chi E^h g + \lambda^{\Delta}(W) \mu^{\Delta}\right] = ((\rho - \Phi)\Lambda_1 - (\eta w \kappa e + \alpha)) W. \end{aligned} \tag{A.7}$$

As the arrival rate for outbursts $\lambda^{\Delta}(W)$ is a constant from (7) for all stress levels, we treat it as a constant here.

The Bellman equation (A.7) holds if two conditions are fulfilled simultaneously. The first makes sure that the right-hand side equals zero. This pins down the first parameter of the value function,

$$\Lambda_1 = \frac{\eta w \kappa e + \alpha}{\rho - \Phi}. \tag{A.8}$$

The second makes sure that the left-hand side equals zero and pins down the second parameter Λ_0,

$$\begin{aligned} \rho\Lambda_0 &= \eta w M e - v_0 m^{1+\zeta} + \frac{\eta w \kappa e + \alpha}{\rho - \Phi} \left[\delta_1 m + \lambda^g \chi E^h g + \lambda^{\Delta}(W) \mu^{\Delta}\right] \\ &= \eta w M e - v_0 m^{1+\zeta} + \frac{\eta w \kappa e + \alpha}{\rho - \Phi} \left[\delta_1 m + \lambda^g \chi \left[\mu^h - \mu\right] + \lambda^{\Delta}(W) \mu^{\Delta}\right] \end{aligned} \tag{A.9}$$

where we used

$$E^h g = \mu^h - \mu \tag{A.10}$$

which follows from (1) and the definition of μ^h and μ before (1).

Note that optimal coping from (A.5) with (A.8) is given by

$$(1+\zeta)\, v_0 m^\zeta = \frac{\eta w \kappa e + \alpha}{\rho - \Phi}\delta_1 \Leftrightarrow m = \left(\frac{\eta w \kappa e + \alpha}{\rho - \Phi}\frac{\delta_1}{(1+\zeta)\, v_0}\right)^{1/\zeta}.$$

For the case where the individual is sick and earns no wage, $w = 0$ as discussed after (A.6), coping m^s while sick drops relative to standard coping m to

$$m^s = \left(\frac{\alpha}{\rho - \Phi^s}\frac{\delta_1}{(1+\zeta)\, v_0}\right)^{1/\zeta}.$$

A.1.3 The value function in closed form

When we plug in values for Λ_0 and Λ_1 into our verified guess, we obtain the value of optimal behaviour. We obtain three versions, one for each range of W described after (11),

$$\begin{aligned} V(W) &= \Lambda_0 - \Lambda_1 W \\ &= \left\{ \begin{array}{c} \Lambda_0^{[0,\bar{W}[} - \frac{\eta w \kappa e + \alpha}{\rho - \Phi} W \\ \Lambda_0^{[\bar{W},W^s[} - \frac{\eta w \kappa e + \alpha}{\rho - \Phi} W \\ \Lambda_0^{[W^s,\infty[} - \frac{\alpha}{\rho - \Phi} W \end{array} \right\} \text{ for } W \in \left\{ \begin{array}{c} [0,\bar{W}[\\ [\bar{W},W^s[\\ [W^s,\infty[\end{array} \right. , \end{aligned}$$

where the Λ_0 are versions of the value from (A.9). In detail,

$$\rho\Lambda_0 = \left\{ \begin{array}{c} \eta w M e - v_0 m^{1+\zeta} + \frac{\eta w \kappa e + \alpha}{\rho - \Phi}\left[\delta_1 m + \lambda^g \chi\left[\mu^h - \mu\right]\right] \\ \eta w M e - v_0 m^{1+\zeta} + \frac{\eta w \kappa e + \alpha}{\rho - \Phi}\left[\delta_1 m + \lambda^g \chi\left[\mu^h - \mu\right] + \bar{\lambda}^\Delta \mu^\Delta\right] \\ -v_0\left[m^s\right]^{1+\zeta} + \frac{\alpha}{\rho - \Phi}\left[\delta_1 m + \lambda^g \chi\left[\mu^h - \mu\right] + \bar{\lambda}^\Delta \mu^\Delta\right] \end{array} \right\} \text{ for } W \in \left\{ \begin{array}{c} [0,\bar{W}[\\ [\bar{W},W^s[\\ [W^s,\infty[\end{array} \right. .$$

The value function is discontinuous at $\bar{W}$ with unchanged slope and (apart from a special case) discontinuous at W^s with a kink. The discontinuity at $\bar{W}$ results from the occurrence of outbursts. The discontinuity at W^s results from the drop of p to p^s and the kink is due to the change in coping intensity.

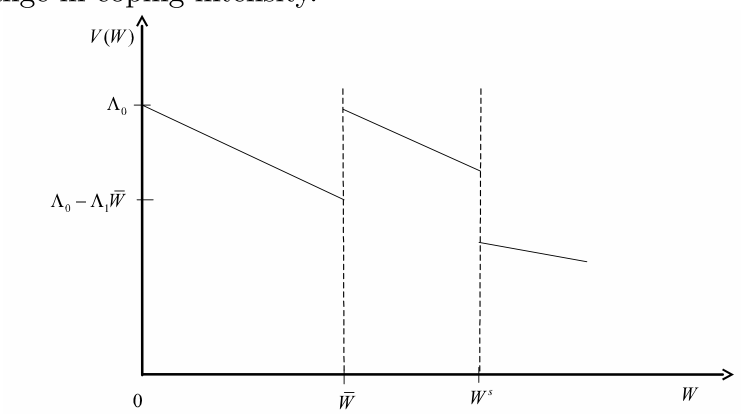

Figure 8 *The value function as a function of stress W with the tolerance level $\bar{W}$ and the sickness-level W^s*

A.2 Proof of the outburst theorem

Proof of (i): The stress level rises by (15a) if $\Phi W(t) - \delta_1 m > 0$. When $\Phi = \Phi^* = \delta_1 m/\bar{W}$, the deterministic part of (12) is negative for any stress levels $W(t) < \bar{W}$ and zero for $\bar{W}$. Hence, if $\Phi > \Phi^*$, $\dot{W}(t) > 0$ for some $W(t) > 0$ – the individual is stress prone.

Proof of (ii): We understand the role of $W(t)$ by remembering that from (16) and (13), $W^* = \frac{\delta_1 m}{\Phi} = \frac{\delta_1 m}{\phi\frac{p}{b} - \delta_0}$. For a given stress level $W(t)$, we can ask whether W^* is larger or smaller than $W(t)$. Under equality, $W(t) = W^* = \frac{\delta_1 m}{\phi\frac{p}{b} - \delta_0}$. Solving this for $\phi\frac{p}{b}$ in an attempt to draw this into fig. 2, we get

$$\phi\frac{p}{b} = \frac{\delta_1 m}{W(t)} + \delta_0 \Leftrightarrow \Phi = \frac{\delta_1 m}{W(t)}.$$

Hence, when $\Phi > \delta_1 m/W(t)$, W^* is smaller than $W(t)$ and the individual is a bad stabilizer.

A.3 The instantaneous expected change of stress

- The differential equation (23) for the instantaneous expected change

When we want to understand how stress changes in expectation for a given stress level $W(t)$, we can proceed as in the derivation of the Bellman equation in app. A.1.1. Instead of computing $E_t dV(W(t))$ for (A.1), we compute $E_t df(W(t))$ for some general function $f(.)$ given the stochastic differential equation for $W(t)$ from (12). Formally, we compute the infinitesimal generator as presented e.g. in Protter (1995, ex. V.7), for many applications, see Wälde (2012, ch. 10.2). We get

$$\begin{aligned} E_t df(W(t)) = {} & f'(W(t))\left\{\phi\frac{p}{b}W(t) - \delta_0 W(t) - \delta_1 m(t)\right\} dt \\ & + E^h\{f(W(t) - \chi g(t)) - f(W(t))\}\lambda^g dt \\ & + E^\Delta\{f(W(t) - \Delta(t)) - f(W(t))\}\lambda^\Delta(W(t))\,dt, \end{aligned}$$

where the analogy to (A.2) is obvious. Specifying $f(x) = x$ to be the identity function and dividing by dt we get

$$\begin{aligned} \frac{E_t dW(t)}{dt} &= \Phi W(t) - \delta_1 m(t) - \lambda^g \chi E^h g(t) - \lambda^\Delta(W(t))\,E^\Delta \Delta(t) \\ &= \Phi W(t) - \delta_1 m(t) - \lambda^g \chi\left[\mu^h - \mu\right] - \lambda^\Delta(W(t))\,E^\Delta \Delta(t) \end{aligned}$$

where the second equality used (A.10). When we take the different stress regions from (12) into account with the corresponding optimal coping levels from (14) and we define $\Omega \equiv \delta_1 m + \lambda^g \chi\left[\mu^h - \mu\right]$ and $\Omega^s \equiv \delta_1 m^s + \lambda^g \chi\left[\mu^h - \mu\right]$ as in the main text after (23), we obtain

$$\frac{E_t dW(t)}{dt} = \left\{\begin{array}{l} \Phi\ W(t) - \Omega \\ \Phi\ W(t) - \Omega - \bar{\lambda}^\Delta \mu^\Delta \\ \Phi^s W(t) - \Omega^s - \bar{\lambda}^\Delta \mu^\Delta \end{array}\right\} \text{ for } \left\{\begin{array}{c} 0 < W(t) \le \bar{W} \\ \bar{W} < W(t) < W^s \\ W^s \le W(t) \end{array}\right. ,$$

where μ^Δ is the mean of $\Delta(t)$ as defined after (5).[57]

[57] I am deeply indebted to Matthias Birkner from the Institute for Mathematics at the Johannes Gutenberg University Mainz for explaining the more subtle issues related to these applications to me. See Birkner et al. (2017) for work in progress studying an analytical description of the density of stress building on forward Kolmogorov equations.

- Does stress fall?

Under which conditions can we expect stress to fall? Focusing on $0 < W(t) \leq \bar{W}$ to start with, the answer is given by

$$\frac{E_t dW(t)}{dt} < 0 \Leftrightarrow \Phi W(t) < \Omega.$$

To understand this condition in detail, one should distinguish four cases: $\Phi \gtrless \Phi^{**}$ and $\Omega \gtrless 0$ where the sign of Φ^{**} from (24) is given by the sign of Ω defined after (23). We can distinguish these four cases analytically as follows,

$$\text{case} \left\{ \begin{array}{c} \text{a} \\ \text{b} \\ \text{c} \\ \text{d} \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \Omega > 0 \\ \Omega < 0 \end{array} \right\} \text{ and } \Phi \text{ sufficiently} \left\{ \begin{array}{l} \text{small, i.e. } \Phi \leq \Phi^{**} \text{ where } \Phi^{**} > 0 \\ \text{large, i.e. } \Phi > \Phi^{**} \text{ where } \Phi^{**} > 0 \\ \text{small, i.e. } \Phi < \Phi^{**} \text{ where } \Phi^{**} < 0 \\ \text{large, i.e. } \Phi \geq \Phi^{**} \text{ where } \Phi^{**} < 0 \end{array} \right. .$$

The four cases are illustrated in the following figure.

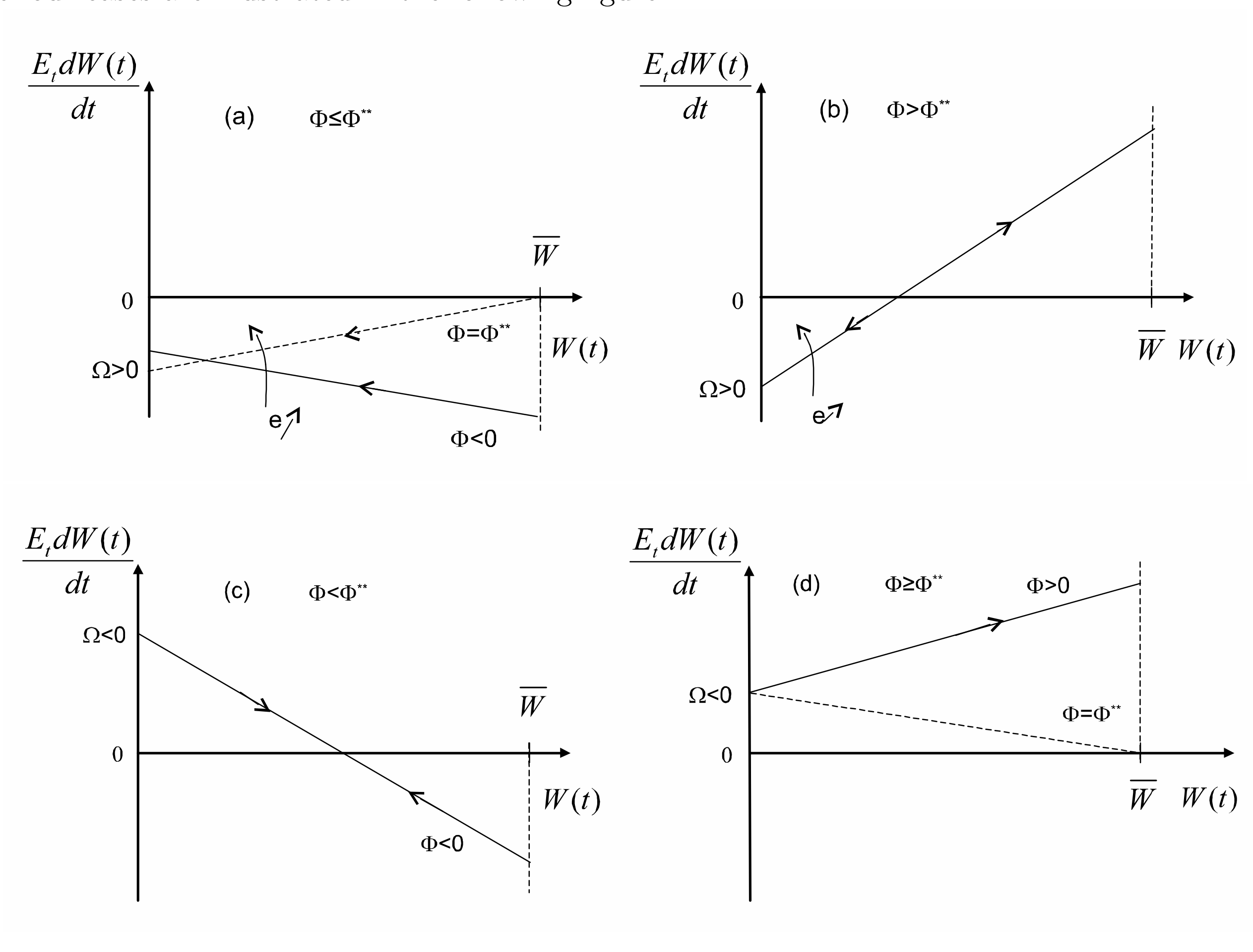

Figure 9 *The expected change of stress for different signs of* Φ *and* Ω *(case a and b covered in main text)*

It seems natural to consider $\Omega > 0$ to represent the "normal" case. As $\delta_1 m > 0$, the second term in the definition of Ω after (23) needs to be strongly negative to make Ω negative. Even when the individual under consideration is mildly overconfident, i.e. subjective expectations exceed objective ones, $\mu > \mu^h$, Ω would still remain positive. This is the reason for focusing on $\Omega > 0$ in the main text.

The analysis of $W(t) > \bar{W}$ proceeds in analogy and is discussed jointly with fig. 6 in the main text.

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