Johannes Gutenberg University Mainz
Graduate School of Economics, Finance, and Management

Advanced Macroeconomic Theory 1 (Part 2)

2018/2019 winter term

Klaus Wälde (lecture) and Jean Roch Donsimoni (tutorial)

Contents

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1 Introduction

1.1 What is macroeconomics?

- Economic growth
- Business cycles
- Employment and unemployment
- Wealth distributions and redistribution
- Money, nominal rigidities and inflation
- Central banks
- Savings and wealth distributions
- Macro and finance
- Fiscal policy
- Behavioural macro
- and much more...

1.2 Who covers what?

- Economic growth
- Business cycles ⇒ Mirko Wiederholt
- Employment and unemployment
- Wealth distributions and redistribution
- Money, nominal rigidities and inflation ⇒ Mirko Wiederholt
- Central banks
- Savings and wealth distributions ⇒ Nicola Fuchs Schündeln
- Macro and finance ⇒ Michalis Haliassos
- Fiscal policy ⇒ Nicola Fuchs Schündeln
- Behavioural macro
- and much more ...

1.3 Who covers what?

- **Economic growth** (our part I)
- Business cycles $\Rightarrow$ Mirko Wiederholt
- **Employment and unemployment** (our part II)
- **Wealth distributions and redistribution**
- Money, nominal rigidities and inflation $\Rightarrow$ Mirko Wiederholt
- Central banks
- Savings and wealth distributions $\Rightarrow$ Nicola Fuchs Schündeln
- Macro and finance $\Rightarrow$ Michalis Haliassos
- Fiscal policy $\Rightarrow$ Nicola Fuchs Schündeln
- **Behavioural macro**
- and much more ...

Part I
Economic growth

2 The convergence debate

2.1 Is there convergence?

- Question: Is there convergence of income per capita over time?
- Are poor countries catching up?

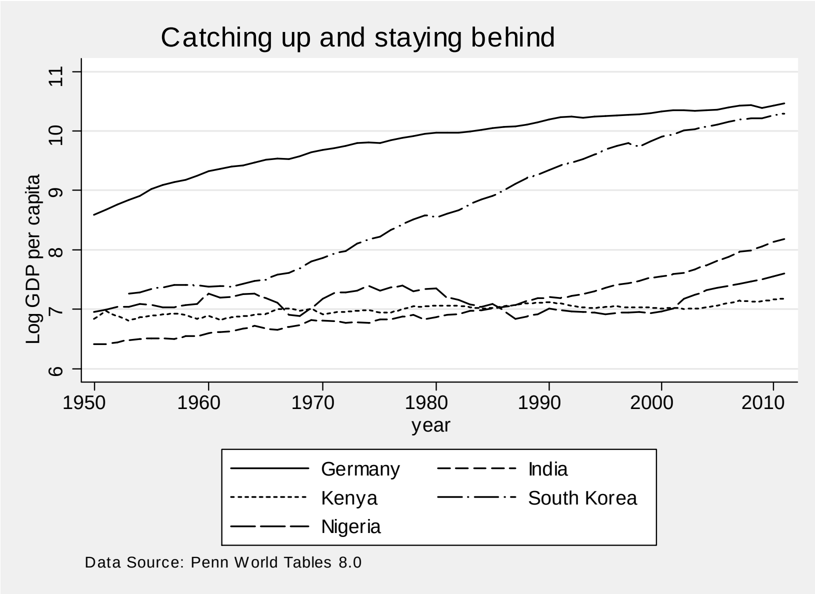


Figure 1 *Catching up and staying behind of some countries*

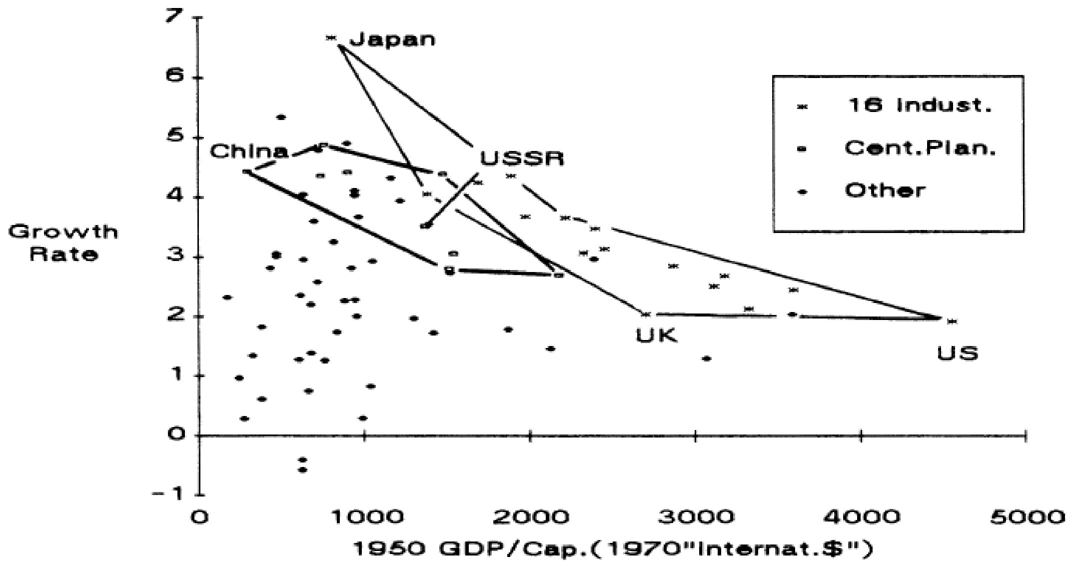


Figure 2 *Growth rate, 1950 - 1980, Gross domestic product per capita vs. 1950 level, 72 countries (Baumol, 1986, fig. 3)*

- Summary of Baumol findings
  - There is convergence of GDP per capita among industrialized countries
  - Convergence not so clear for centrally planned economies
  - No convergence for less developed countries
- Big subsequent discussion on convergence or not
  - View in 2006 (see introduction in Sala-i-Martin, 2006): no convergence (see fig. Ia below)
  - Once population weights are used, this result disappears (see fig. Ib below)
  - Hence, evidence for convergence reappears for the entire sample of countries

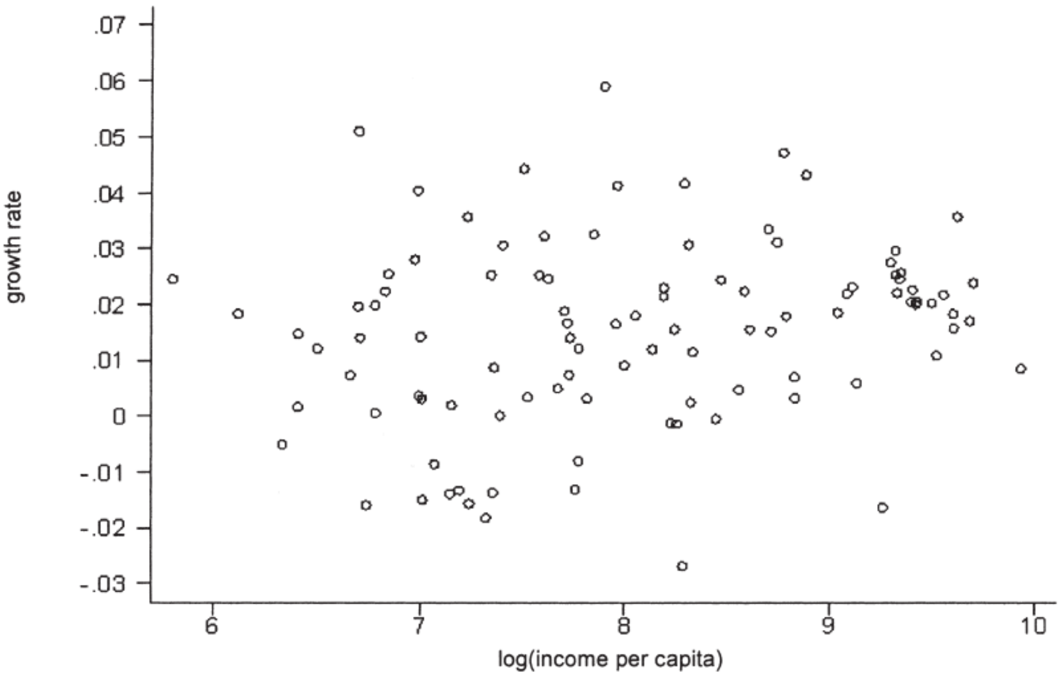


Figure 3 *Growth vs. initial income (unweighted) (Sala-i-Martin, 2006, fig. Ia)*

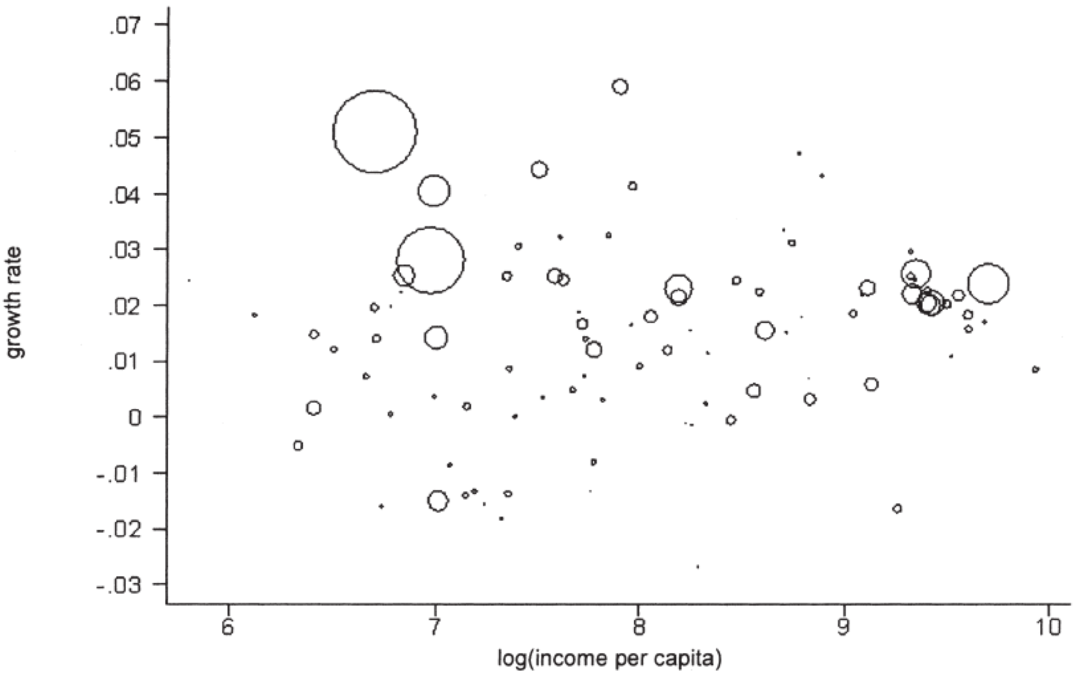


Figure 4 *Growth vs. initial income (Population-Weighted) (Sala-i-Martin, 2006, fig. Ib)*

2.2 Questions for economic theory

- Why are some countries richer than others?
- Why do countries grow?
- Why do some countries grow faster than others?
- Can countries grow faster only temporarily or also permanently?

3 Neoclassical growth theory

3.1 Some background

- Exogenous saving rate
  - Solow (1956)
- Optimal saving
  - Cass (1965) and Koopmans (1965)
  - Ramsey (1928)
- Textbooks (examples)
  - Aghion and Howitt (1998), "Endogenous growth theory", ch 1.1 and 1.2
  - Barro and Sala-i-Martin (2004) "Economic Growth"
  - Wälde (2012), "Applied Intertemporal Optimization"
    (pdf-download at www.waelde.com/aio)

3.2 The Solow-Cass-Koopmans-Ramsey model

3.2.1 The Solow model

- Starting point: Solow (1965) growth model

  - Capital stock $K(t)$ follows

    $$\dot{K}(t) \equiv \frac{dK(t)}{dt} = sK(t)^{\alpha}[A(t)L(t)]^{1-\alpha} - \delta K(t)$$

    where s is exogenous saving rate, α is output elasticity of capital and δ is depreciation rate

  - Population $L(t)$ and labour productivity $A(t)$ grow at rates n and g,

    $$L(t) = L_0 e^{nt}, \quad A(t) = A_0 e^{gt}$$

- What are the central findings of the Solow model?

  - see next page
  - (see above textbooks for details and derivations)

- Why do countries grow?
  - What does growth mean? Growth of GDP, GDP per capita or something else (happiness?)?
  - GDP: because of population growth and TFP growth
  - GDP per capita: because of TFP growth (technological progress)
  - Is this a non-explanation for long-run growth? Yes - TFP growth is exogenous!
- Why are some countries richer than others (in terms of GDP per capita)?
  - in the short run: more capital, higher TFP level
  - in the long run: conditional convergence - the long-run capital stock $k^* \equiv K/(AL)$ depends on parameters (e.g. saving rate, depreciation). If they differ, GDP per capita differs in the long run
- Why do some countries grow faster than others?
  - temporarily: there can be a catching-up process
  - in the long run: we do not know (given the Solow model)

3.2.2 The issue of the optimal saving rate

- Question: What are determinants of saving rate and why should we want to explain it?
  - There are large empirical differences across individuals, countries and over time
  - Theoretical curiosity: A central variable should not be left unexplained

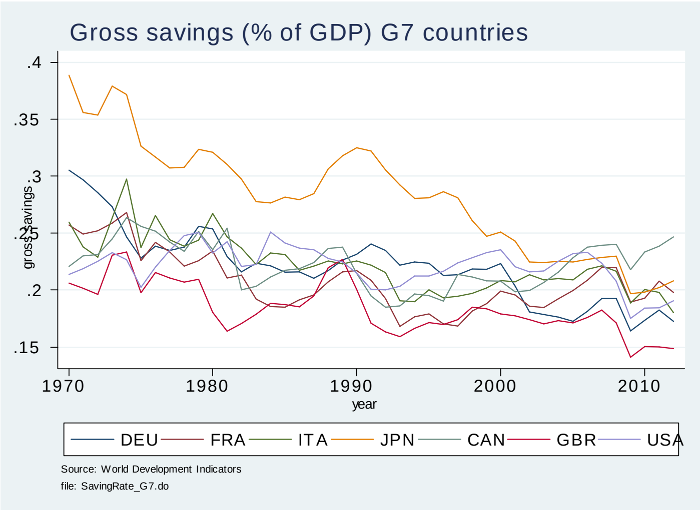


Figure 5 *Savings rates differ across countries and over time*

- Approach
  - Belief of (most) economists: Consumption and saving (and therefore saving rate) are optimally chosen
  - So – let us construct a maximization problem which explains saving rate endogenously
- Technical background
  - Wälde (2012, ch 5.6.3)

3.2.3 The model and optimal behaviour

- Preferences
  - We study a central planner problem
  - This allows to focus on optimally chosen s in the simplest way
  - Objective function is social welfare function $U(t)$

$$U(t) = \int_t^\infty e^{-\rho[\tau - t]} u(C(\tau))\, d\tau \tag{3.1}$$

where

  - $u(C(\tau))$ is the instantaneous utility function,
  - $\rho > 0$ is the time preference rate (measuring impatience of individual) and
  - $e^{-\rho[\tau - t]}$ is the discount factor (or function)
  - Planning/ optimal behaviour starts in t (like 'today') and goes up to infinity ∞

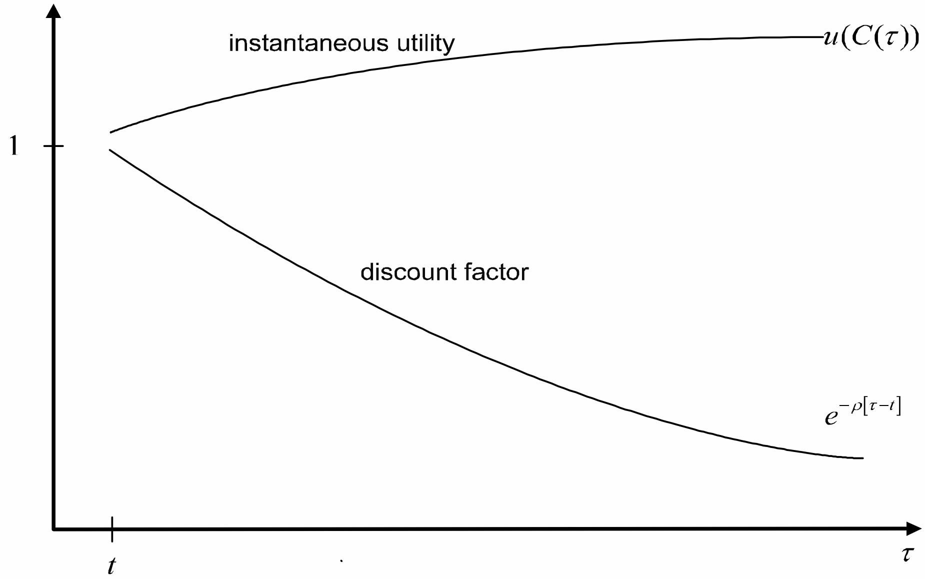


Figure 6 *The discount factor* $e^{-\rho[\tau-t]}$ *and (an example of) instantaneous utility as a function of time* τ

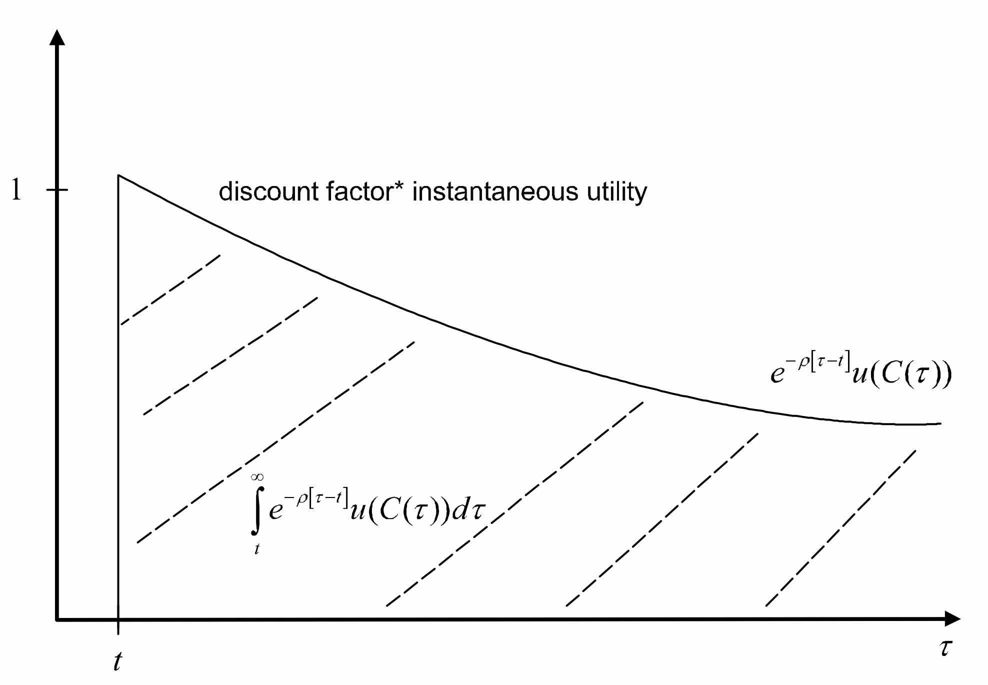


Figure 7 *The objective function is shown by the shaded area (and is maximized by the choice of the consumption path $C(\tau)$). Note that the function $e^{-\rho[\tau-t]}u(C(\tau))$ must fall (for boundedness reasons - see Wälde, 2012, ch. 5.3.2)*

- Instantaneous utility
  - Functional form ...

$$u(C) = \left\{ \begin{array}{c} \frac{C^{1-\sigma}-1}{1-\sigma} \\ \ln C \end{array} \right\} \text{for} \left\{ \begin{array}{l} \sigma \neq 1 \text{ and } \sigma > 0 \\ \sigma = 1 \end{array} \right. \tag{3.2}$$

  - ... implies constant intertemporal elasticity of substitution of $1/\sigma$
- The resource constraint
  - Maximization problem becomes meaningful only with a constraint
  - Constraint here is a resource constraint as we look at the economy as a whole
  - Capital evolves according to

$$\dot{K}(t) = Y(K(t), L) - \delta K(t) - C(t) \tag{3.3}$$

  - The change in the capital stock (net investment) is given by output minus consumption (gross investment) minus depreciation $\delta K(t)$

- Maximization problem
  - The planner chooses the path $\{C(t)\}$ of consumption $C(t)$ between t and infinity to maximize $U(t)$ from (3.1) subject to the constraint (3.3)
  - Optimal saving rate is only implicitly determined
- The Keynes-Ramsey rule
  - The Keynes-Ramsey rule reads (see sect. 3.2.4, sect. 3.2.5 and Exercise 6.1.1)

$$-\frac{u''(C(t))}{u'(C(t))}\dot{C}(t) = \frac{\partial Y(K(t), L)}{\partial K(t)} - \delta - \rho$$

  - This is one equation fixing the optimal path of consumption over time – joint with the equation for capital in (3.3)
  - We therefore ended up with a two-dimensional differential equation system for two variables, $C(t)$ and $K(t)$
  - This is the end of the maximization problem fixing optimal consumption and (implicitly) optimal saving rates (assuming for the time being two boundary conditions)

- How can we understand the Keynes-Ramsey rule?
  - The term $-\frac{u''(C(t))}{u'(C(t))}$ is the Arrow-Pratt measure of absolute risk aversion. It measures (i) the curvature of the instantaneous utility function, i.e. (ii) how much individuals dislike risk
  - More capital $K(t)$ increases output by the marginal productivity of capital, $\frac{\partial Y(K(t),L)}{\partial K(t)}$, which can be called gross interest rate
  - Subtracting the depreciation rate δ gives the net interest rate or net return to an additional unit of capital
  - Consumption grows ($\dot{C}(t) > 0$) if and only if the right-hand side is positive. This is the case when net return from more capital is larger than the time preference rate
  - This difference captures the trade-off between the reward to less consumption today (the net return to an additional unit of capital) and the downside/ punishment/ disadvantage of less consumption today (which is captured by the time preference rate, i.e. by the discounting of consumption that is shifted to the future)

- The specific Keynes-Ramsey rule
  - For our instantaneous CES utility function from (3.2), the rule reads

$$\frac{\dot{C}(t)}{C(t)} = \frac{\frac{\partial Y(K(t),L)}{\partial K(t)} - \delta - \rho}{\sigma} \tag{3.4}$$

  - This illustrates, as before the trade-off between the net return $\frac{\partial Y(K(t),L)}{\partial K(t)} - \delta$ and the time-preference rate ρ
  - This tells us, given that $1/\sigma$ is the intertemporal elasticity of substitution, that ...
  - ... an individual with a high $1/\sigma$ reacts more strongly (in terms of changes in consumption growth) to changes in returns to capital than an individual with a low intertemporal elasticity of substitution

- A central planner that chooses consumption and thereby the savings rate optimally follows a consumption path that satisfies this Keynes-Ramsey rule

3.2.4 How to obtain Keynes-Ramsey-Rules: Hamiltonians [background]

- Consider a central planner who maximises a social welfare function (3.1)

$$\max_{\{C(\tau)\}} \int_t^\infty e^{-\rho[\tau-t]} u\left(C\left(\tau\right)\right) d\tau$$

subject to a resource constraint

$$\dot{K}\left(t\right) = Y\left(K\left(t\right), L\right) - \delta K\left(t\right) - C\left(t\right) \tag{3.5}$$

where the instantaneous utility function $u\left(C\left(\tau\right)\right)$ is given as

$$u\left(C\right) = \frac{C^{1-\sigma} - 1}{1-\sigma} \tag{3.6}$$

- The *current* value Hamiltonian, consisting of instantaneous utility plus $\lambda\left(t\right)$ multiplied by the relevant part of the constraint, is written as

$$H = u\left(C\left(t\right)\right) + \lambda\left(t\right)\left[Y\left(K\left(t\right), L\right) - \delta K\left(t\right) - C\left(t\right)\right]$$

(for the present value Hamiltonian, see Wälde, 2012, ch. 5.7)

- Optimality conditions are

$$u'(C(t)) = \lambda(t), \tag{3.7}$$

$$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial H}{\partial K} = \rho\lambda(t) - \lambda(t)\left[Y_K(K(t), L) - \delta\right] \tag{3.8}$$

- Differentiating the first-order condition (3.7) with respect to time gives $u''(C(t))\dot{C}(t) = \dot{\lambda}(t)$. Inserting this and (3.7) into the second condition again gives, after some rearranging,

$$-\frac{u''(C(t))}{u'(C(t))}\dot{C}(t) = Y_K(K(t), L) - \delta - \rho$$

- Given the instantaneous utility function (3.6), we find $-u''(C(t))/u'(C(t)) = \sigma/C(t)$
- Thus, the Keynes-Ramsey rule reads

$$\frac{\dot{C}(t)}{C(t)} = \frac{Y_K(K(t), L) - \delta - \rho}{\sigma} \tag{3.9}$$

3.2.5 How to obtain Keynes-Ramsey-Rules: Dynamic Programming [background]

- We now consider a similar maximization problem
- Consider an individual choosing a path of consumption $\{c(\tau)\}$ to maximise her utility

$$U(t) = \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\, d\tau \tag{3.10}$$

  subject to a budget constraint

$$\dot{a}(t) = r(t)\, a(t) + w(t) - p(t)\, c(t) \tag{3.11}$$

- We introduce a value function $V(a(t))$ of the optimal program that is defined
  - by the maximum overall utility level that can be reached
  - by choosing the consumption path optimally,
  - given the constraint,

$$V(a(t)) \equiv \max_{\{c(\tau)\}} U(t) \text{ subject to (3.11)}$$

- When households behave optimally between today and infinity by choosing the optimal consumption path $\{c(\tau)\}$, their overall utility $U(t)$ is given by $V(a(t))$

- A prelude on the Bellman equation
  - The Bellman equation for optimisation problems of the above type reads

$$\rho V\left(a\left(t\right)\right)=\max_{c(t)}\left\{u\left(c\left(t\right)\right)+\frac{dV\left(a\left(t\right)\right)}{dt}\right\} \tag{3.12}$$

  - The derivation of the Bellman equation under continuous time is not as obvious as under discrete time (see Wälde, 2012, ch. 3.3.2 and 6.1.2)
  - To get a feeling of its intuitive meaning, consider the following steps
  - Given the objective function in (3.10), we can ask how overall utility $U\left(t\right)$ changes over time
  - To this end, compute the derivative $dU\left(t\right)/dt$ (by employing the Leibniz rule)

- The Leibniz rule
  - When computing derivatives of functions that are or include integrals, the following rule is useful (see Wälde, 2012, ch. 4.3.1)
  - Consider a function $z(x)$ with argument x, defined by the integral

$$z(x) \equiv \int_{a(x)}^{b(x)} f(x, y) \, dy,$$

    where $a(x)$, $b(x)$ and $f(x, y)$ are differentiable functions
  - Note that x is the only argument of z, as y is integrated out on the right-hand side
  - The Leibniz rule says

$$\frac{d}{dx} z(x) = b'(x) f(x, b(x)) - a'(x) f(x, a(x)) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, y) \, dy.$$

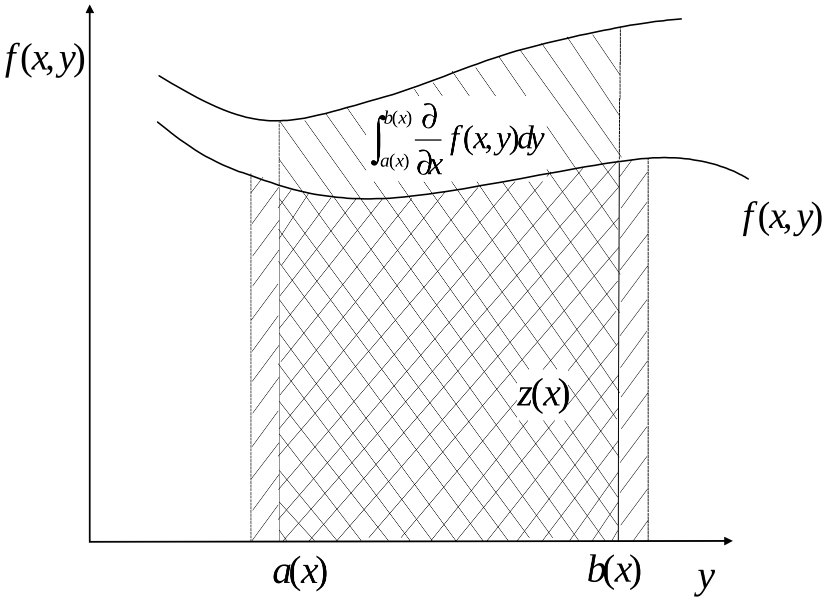


Figure 8 *Illustration of the Leibniz rule*

- A prelude on the Bellman equation (cont'd)
  - ... to this end, compute the derivative $dU(t)/dt$ and find (see also exercise 6.1.3)

$$\dot{U}(t) = -e^{-\rho[t-t]}u(c(t)) + \int_t^\infty \frac{d}{dt}e^{-\rho[\tau-t]}u(c(\tau))\,d\tau = -u(c(t)) + \rho U(t).$$

  - Rearranging this equation gives $\rho U(t) = u(c(t)) + \dot{U}(t)$
  - When overall utility is replaced by the value function, we obtain $\rho V(a(t)) = u(c(t)) + \dot{V}(a(t))$ which corresponds in its structure to the Bellman equation (3.12)

- Maximization by dynamic programming
  - The (appropriately modified) Bellman equation (3.12) is the starting point
  - Dynamic programming can then be presented by going through three steps

- DP1: Bellman equation and first-order conditions
  - First we compute $dV(a(t))/dt = V'(a(t))\dot{a}$ needed in (3.12)
  - The Bellman equation (3.12) can then be rewritten, using the budget constraint (3.11), as
$$\rho V(a(t)) = \max_{c(t)} \{u(c(t)) + V'(a(t))[ra + w - pc]\} \quad (3.13)$$
  - The first-order condition reads
$$u'(c(t)) = pV'(a(t)) \quad (3.14)$$
    and makes consumption a function of the state variable, $c(t) = c(a(t))$
  - The trade-off is easy to see
    * When consumption goes up today by one unit, wealth goes down by p units
    * Higher consumption increases utility by $u'(c(t))$, p units less wealth reduces overall utility by $pV'(a(t))$
  - [A remark on research strategy]
    * We could stop with analytical steps here
    * We could solve two equations (3.13) and (3.14) for control variable $c(a)$ and value function $V(a)$ numerically
    * Further analytical steps yield more economic insights, however

- DP2: Evolution of the costate variable
  - The maximized Bellman equation is given by
$$\rho V(a) = u(c(a)) + V'(a)[ra + w - pc(a)].$$
  - Computing the derivative with respect to $a(t)$ using the envelope theorem (see Exercise 6.1.3) gives an expression for the shadow price of wealth
$$\begin{aligned}\rho V'(a) &= V''(a)[ra + w - pc] + V'(a) r \Leftrightarrow \\ (\rho - r) V'(a) &= V''(a)[ra + w - pc].\end{aligned} \tag{3.15}$$
  - Computing the derivative of the costate variable $V'(a)$ with respect to time gives
$$\frac{dV'(a)}{dt} = V''(a)\dot{a} = (\rho - r) V'(a),$$
where the last equality used (3.15)
  - Dividing by $V'(a)$ and using the usual notation $\dot{V}'(a) \equiv dV'(a)/dt$, this can be written as
$$\frac{\dot{V}'(a)}{V'(a)} = \rho - r. \tag{3.16}$$
  - This equation describes the evolution of the costate variable $V'(a)$, the shadow price of wealth

- DP3: Inserting first-order conditions
  - The derivative of the first-order condition with respect to time is given by (apply first logs)

$$\frac{u''(c)}{u'(c)}\dot{c} = \frac{\dot{p}}{p} + \frac{\dot{V}'(a)}{V'(a)}.$$

  - Inserting (3.16) gives

$$\frac{u''(c)}{u'(c)}\dot{c} = \frac{\dot{p}}{p} + \rho - r \Leftrightarrow -\frac{u''(c)}{u'(c)}\dot{c} = r - \frac{\dot{p}}{p} - \rho.$$

  - This is the well-known Keynes Ramsey rule

3.2.6 Comparing dynamic programming to Hamiltonians [background]

- Consider the "additional" optimality condition (3.8) in the Hamiltonian approach

$$\dot{\lambda}(t) = \rho\lambda(t) - \lambda(t)\left[Y_K(K(t), L) - \delta\right] \Leftrightarrow \frac{\dot{\lambda}(t)}{\lambda(t)} = \rho - \left[Y_K(K(t), L) - \delta\right]$$

- Compare it to condition (3.16) from dynamic programming

$$\frac{\dot{V}'(a)}{V'(a)} = \rho - r$$

- Both conditions need to hold when behaviour is to be optimal
- This suggests the obvious interpretation

$$V'(a) = \lambda$$

- This is where the interpretation for the costate variable as a shadow price in the Hamiltonian approach came from
  - The costate variable $\lambda(t)$ stands for the increase in the value of the optimal program when an additional unit of the state variable becomes available
  - Hence, the interpretation of a costate variable $\lambda(t)$ is similar to the interpretation of the Lagrange multiplier in static maximization problems – a shadow price

3.3 A phase diagram analysis

Let us return to our optimal saving problem from section 3.2.3

- How to proceed from the Keynes-Ramsey rule?
  - We need to analyse optimal consumption (3.4) jointly with the evolution of the capital stock as described in the resource constraint (3.3)
  - Technically speaking, we face a two-dimensional differential equation system (which is non-linear)
  - (see Exercise 6.1.5 for a one-dimensional-differential-equation-system example)
  - Qualitative method to understand its properties: Phase diagram analysis (see Wälde, 2012, ch. 4.2 for more background)

- First step: Find a steady state (see Exercise 6.1.5, question 1, for a general definition of a steady state)
  - Definition of steady state here: Values of K^* and C^* for which the capital stock and consumption do not change over time
  - Formally

$$\dot{K}(t) = 0 \Leftrightarrow C^* = Y(K^*, L) - \delta K^* \tag{3.17}$$

$$\dot{C}(t) = 0 \Leftrightarrow \frac{\partial Y(K^*, L)}{\partial K^*} - \delta = \rho \tag{3.18}$$

  - These two (algebraic) equations pin down K^* and C^*
  - We can plot them into a phase diagram, see figure 9 below

- Second step: How do $K(t)$ and $C(t)$ change when they are not in the steady state?
- Step 2a: Draw the zero-motion lines (curves on which consumption or capital do not change)
  - Zero motion line for capital comes from (3.17)
  - Zero motion line for consumption is a vertical line given by (3.18)
  - steady state is NOT at the point where consumption is highest (compare 'golden rule')

- Step 2b: Draw 'arrows of motion' into the phase diagram
  - Starting with the resource constraint, we know

$$\dot{K}(t) \geq 0 \Leftrightarrow Y(K(t), L) - \delta K(t) \geq C(t)$$

  - In words: Capital rises whenever consumption is below the zero motion line
  - This is intuitive: capital rises if output minus depreciation is larger than consumption
  - Capital falls above the zero-motion line, capital is 'eaten up'
  - The Keynes-Ramsey rule tells us

$$\dot{C}(t) \geq 0 \Leftrightarrow \frac{\partial Y(K(t), L)}{\partial K(t)} - \delta \geq \rho$$

  - Consumption rises if net return is sufficiently large, or if individuals are sufficiently patient
  - Consumption rises whenever we are to the left of the zero motion line for consumption, i.e. for $K(t) < K^*$ (as for $K(t) < K^*$, the marginal productivity is larger than at K^* given concavity of the production function)
  - consumption falls whenever we are at $K(t) > K^*$
  - Draw all of this into the phase diagram in figure 9 as well

- Step 2c: Draw trajectories into phase diagram
  - Starting at any point in the phase diagram, look at arrows of motion and describe changes over time by arrows on trajectories
  - Here, we can find a saddle path that leads to the steady state (which is a saddle point here)

- How does optimal consumption evolve over time?
  - Start with some initial (exogenous) capital stock K_0
  - Optimal consumption level is given by the consumption level that puts the economy on the saddle path. In the figure, this is C_0
  - As of then, the economy grows and approaches the steady state

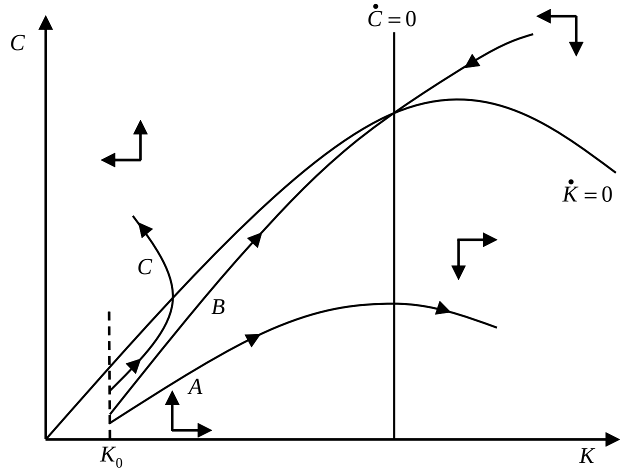


Figure 9 *Phase diagram analysis for optimal consumption of a central planner*

3.4 More background on phase diagrams [background]

(not covered in lecture – important for life, not important for exam)
(for more detail, see Wälde, 2012, ch.4)

- Types of fixpoints

Definition 1 *A fixpoint is called a*

$$\left.\begin{array}{c}\textit{center}\\ \textit{saddle point}\\ \textit{focus}\\ \textit{node}\end{array}\right\} \Leftrightarrow \left\{\begin{array}{c}\textit{zero}\\ \textit{two}\\ \textit{all}\\ \textit{all}\end{array}\right\} \textit{trajectories pass through the fixpoint}$$

$$\textit{and} \left\{\begin{array}{l} \\ \\ \\ \textit{on} \left\{\begin{array}{c}\textit{at least one trajectory, both variables are non-monotonic}\\ \textit{all trajectories, one or both variables are monotonic}\end{array}\right.\end{array}\right.$$

A node and a focus can be either stable or unstable.

- A node

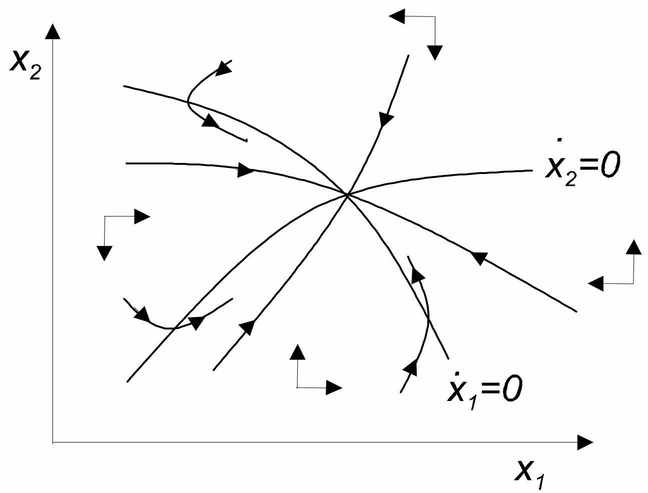


Figure 10 *Phase diagram for a stable node*

- A node is a fixpoint through which all trajectories go and where the time paths implied by trajectories are monotonic for at least one variable

• A focus

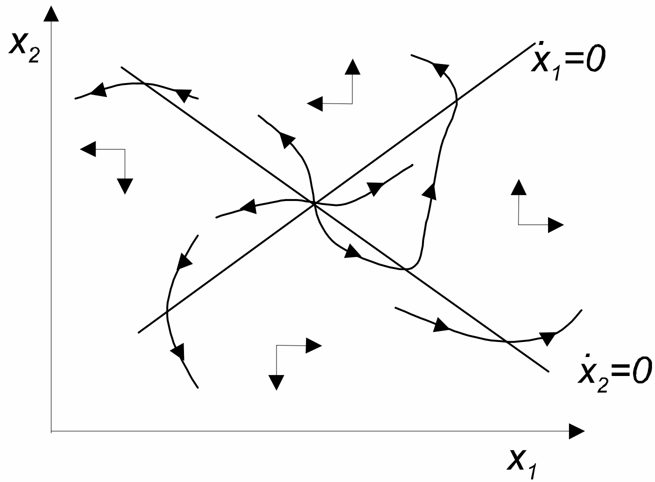


Figure 11 *Phase diagram for a focus*

• A phase diagram with a focus looks similar to one with a node. The difference lies in the non-monotonic paths of the trajectories. As drawn here, x_1 or x_2 first increase and then decrease on some trajectories.

- A center

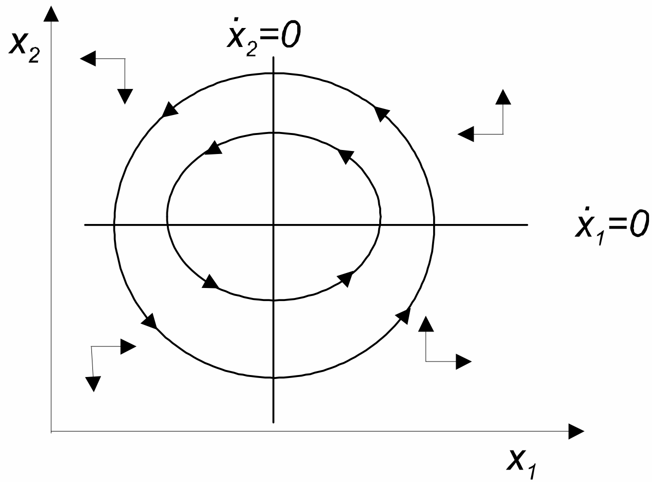


Figure 12 *Phase diagram for a center*

- very special case, rarely found in models with optimizing agents
- Standard example is the predator-prey model (see Exercise 6.1.5)

- Limitations
  - A phase diagram analysis allows us to identify a saddle point. But if no saddle point can be identified, it is generally not possible to distinguish between a node, focus or center.
  - In the linear case, more can be deduced from a graphical analysis. This is generally not necessary, however, as there is a closed-form solution. The definition of various types of fixpoints is then based on Eigenvalues of the system.
- Multidimensional systems
  - If we have higher-dimensional problems where $x \in R^n$ and $n > 2$, phase diagrams are obviously difficult to draw
  - In the three-dimensional case, plotting zero motion surfaces sometimes helps to gain some intuition. A graphical solution will generally, however, not allow us to identify equilibrium properties like saddle-path or saddle-plane behaviour.

3.5 What have we learned?

- Solow growth model
  - Reminder of central economic results
  - Issue with Solow model: exogenous saving rate
- What are the determinants of the optimal saving rate?
  - Instantaneous and intertemporal utility function needed
  - New parameters: intertemporal elasticity of substitution $1/\sigma$ and time preference rate ρ
  - Determinants are (inter alia) σ and ρ

- How can optimal saving rates and optimal consumption rules be derived and understood?
  - Hamiltonian
  - Dynamic programming
  - Phase diagram analysis useful qualitative tool
- Further issue with Solow growth model
  - Why does GDP per capita grow in the long run?
  - More theory needed ...

4 New growth theory: Incremental innovations

4.1 Some background on the "new" endogenous growth theory

Innovation and growth - main contributions and ideas

- Romer (1986)
  - The use of capital requires knowledge which is a public good
  - Capital accumulation goes hand in hand with knowledge accumulation
  - Technically, this implies constant returns to scale in factors of production that can be accumulated
  - Assume the production function reads $Y(t) = AK(t)$
  - The marginal productivity of capital is constant (and given by A), growth never comes to an end (even in the absence of technological progress)

- Romer (1990) "Endogenous Technological Change"
  - Presents an economic mechanism highlighting the economics behind innovation and growth
  - Again, knowledge arises as an externality in the process of innovation, of intentional R&D
  - Technically, constant returns in the R&D process imply a constant long-run growth rate
- Aghion and Howitt (1992) "A model of growth through creative destruction"
  - Innovations are no longer incremental but can have negative side-effects for competitors
  - Schumpetarian creative-destruction view of the growth process
  - Also highlights the downsides of technological progress

Further developments

- Non-scale growth models
  - Jones (1995)
  - Segerstrom (1998)
- Unified growth theory
  - Galor (2005) provides a survey
  - Literature studies the question of how an economy moves from subsistence activities to systematic economic growth
- International trade and economic growth
  - Grossman and Helpman (1991)
  - Many others
- International trade, unemployment and inequality
  - Helpman, Itskhoki and Redding (2010)
  - Helpman and Itskhoki (2010)

4.2 The principle of endogenous growth theory

- The idea
  - An economy needs resources for innovation and growth
  - Technological progress does not come without cost (as in Solow growth model)
  - Idea goes back at least to Shell (1966)

- A planner setup
  - Social welfare function reads

$$U(t) = \int_t^\infty e^{-\rho[\tau - t]} u(C(\tau))\, d\tau$$

where utility from aggregate consumption is given by

$$u(C(\tau)) = \frac{C(\tau)^{1-\sigma} - 1}{1-\sigma}, \ \sigma > 0, \sigma \neq 1$$

  - The objective function is maximized subject to two constraints

$$C(\tau) = A(\tau)\left[L - L_A(\tau)\right]$$
$$\frac{\dot{A}(\tau)}{A(\tau)} = L_A(\tau)$$

where $A(\tau)$ is labour productivity in $\tau \geq t$ and labour L is the only (fixed) factor of production. The number of workers in the research sector is given by $L_A(\tau)$

- Solution of maximization problem and the principle
  - The central planner chooses $L_A(\tau)$ and faces a classic trade-off
  - Few workers (low $L_A(\tau)$) in the R&D sector imply low growth but high consumption (at least currently). Many workers in R&D imply fast growth but low consumption
  - The principle of endogenous growth theory: some workers in R&D sector are needed. What are the determinants of $L_A(\tau)$?

4.3 The Grossman and Helpman model

- Questions
  - How can one imagine a growth process being driven by R&D?
  - How does "endogenous technological change" in the spirit of Romer (1990) work?
  - What are the determinants of the economy-wide growth rate?
- Approach
  - We follow Grossman and Helpman (1991, ch. 3) due to its conceptional clarity
  - One fundamental problem to solve: How can a firm finance R&D if R&D is costly and only at some future point leads to success?
  - Firms under perfect competition would not work (but see models with "prototypes")
  - Answer: Firm must act under imperfect competition, thereby make profits which allow to repay the costs of R&D

- Technologies
  - We are in a world with one differentiated good – there is not "one car" but many different "varieties of cars"
  - This is known as a "Dixit-Stiglitz framework", going back to Dixit and Stiglitz (1977), see Exercise 6.1.7
  - A variety i is produced by employing labour $l(i,t)$

$$x(i,t) = l(i,t)$$

  - New varieties are developed in an R&D sector, using labour as well

$$\dot{n}(t) = \varphi L_R(t) \tag{4.1}$$

    where
  - $n(t)$ is the current number of varieties
  - φ is labour productivity in the R&D sector
  - $L_R(t)$ is the number of researchers and
  - $\dot{n}(t) \equiv dn(t)/dt$ is the increase in the number of varieties

- Labour market
  - The economy is endowed with L workers (fixed quantity)
  - Workers either work in R&D sector or in production sector, i.e. wage is the same in both sectors in equilibrium
  - Labour market clears if

$$\int_0^n l(i,t)\,di + L_R(t) = L$$

    meaning that total employment in production sector (the integral $\int_0^n l(i,t)\,di$) plus employment $L_R(t)$ in R&D equals supply L

- Household preferences
  - We work with a representative agent r (and ignore distributional issues)
  - Intertemporal (or overall) objective function of the representative household reads

$$U(t) = \int_t^\infty e^{-\rho[\tau - t]} u(c_r(\tau))\, d\tau \tag{4.2}$$

    where $u(c_r(\tau))$ is instantaneous utility from consumption (index) $c_r(\tau)$ in τ
  - Consumption index c_r reflects Dixit-Stiglitz (1977) structure with a continuum of varieties and a "love-of-variety" interpretation

$$c_r(\tau) \equiv \left(\int_0^n c_r(i,\tau)^\theta\, di \right)^{1/\theta}, \quad 0 < \theta < 1 \tag{4.3}$$

  - Elasticity of substitution ε between varieties exceeds unity

$$\varepsilon = \frac{1}{1-\theta} > 1$$

    as otherwise firms (see below) would make profits too easily
  - The instantaneous utility function is logarithmic

$$u(c_r(\tau)) = \ln c_r(\tau) \tag{4.4}$$

- Budget constraint
  - The constraints of the household include a budget constraint (see Exercise 6.1.6 for a derivation)
  $$\dot{a}_r(t) = r(t)\, a_r(t) + w(t) - e_r(t) \tag{4.5}$$
  where the change in wealth $\dot{a}_r$ (of the representative agent) is determined by the difference between capital income ra, labour income w and consumption expenditure e_r, and a constraint/ definition for total expenditure
  $$e_r(t) = \int_0^n p(i,t)\, c_r(i,t)\, di$$
  - What is aggregate wealth $a(t) = a_r(t)\, L$ in this economy?
  $$a(t) = v(t)\, n(t)$$
  Aggregate wealth $a(t)$ is given by the value $v(t)$ of a (representative) firm times the number of firms, which is also the number of varieties, i.e. $n(t)$
  - The number of firms and the number of varieties is the same as
    * Costs of innovation and imitation (not modelled) are the same
    * No incentive to (re) develop an existing variety
    * Each firm develops a new variety.

- What is the interest rate in this setup?

$$r(t) \equiv \frac{\pi(t) + \dot{v}(t)}{v(t)}$$

  The interest rate in the dynamic budget constraint is an "abbreviation" of this longer expression (which can be seen from deriving the budget constraint, see Exercise 6.1.6)

- Profits by firms are denoted by π, the change of the value of a firm is $\dot{v}$. The sum, relative to the price of a firm is the interest rate

4.4 Optimal behaviour

- Households (static)
  - Households behave optimally at each point in time and also over time
  - Optimal choice between varieties leads to instantaneous demand function (see Exercise 6.1.7)

$$c_r(i,t) = \frac{p(i,t)^{-\varepsilon}}{\int_0^{n(t)} p(i,t)^{1-\varepsilon}\, di} e_r(t) \tag{4.6}$$

  - Demand for variety i in (4.6) depends on the prices of all the other varieties as well (unlike in the more standard Cobb-Douglas case).
  - Let us define the price index of all varieties $P(t)$ as

$$P(t) = \int_0^{n(t)} p(i,t)^{1-\varepsilon}\, di \tag{4.7}$$

  - such that optimal consumption of variety i becomes

$$c_r(i,t) = \frac{p(i,t)^{-\varepsilon}}{P(t)} e_r(t) \tag{4.8}$$

- Households (the indirect utility function)
  - Given this optimal instantaneous behaviour, how do we optimally spread total expenditure $E(\tau)$ over time τ?
  - Inserting (4.8) into the consumption index (4.3), where we also note that $\theta = \frac{\varepsilon-1}{\varepsilon}$, gives

$$c_r(\tau) = \left(\int_0^n c_r(i,\tau)^\theta \, di \right)^{\frac{1}{\theta}} = \left(\int_0^n \left(\frac{p(i,\tau)^{-\varepsilon}}{P(\tau)} e_r(\tau) \right)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

  - After some steps (see again Exercise 6.1.7) we get

$$c_r(\tau) = \frac{e_r(\tau)}{P(\tau)^{\frac{1}{1-\varepsilon}}}$$

  - Inserting this result into (4.2), using the functional form (4.4), yields

$$U(t) = \int_t^\infty e^{-\rho(\tau-t)} \ln c_r(\tau) \, d\tau = \int_t^\infty e^{-\rho(\tau-t)} \ln \left(\frac{e_r(\tau)}{P(\tau)^{\frac{1}{1-\varepsilon}}} \right) d\tau \qquad (4.9)$$

  - This is the indirect intertemporal utility function

- Households (dynamic)
  - Define $E(\tau)$ as total expenditure in this economy, i.e. $E(\tau) = e_r(\tau) L$
  - Now choose $E(\tau)$ to maximise indirect utility function (4.9) subject to budget constraint (4.5)
  - The Hamiltonian for our problem reads

$$\begin{aligned}\mathbb{H}(\tau) &= \ln \frac{e_r(\tau)}{P(\tau)^{\frac{1}{1-\varepsilon}}} + \lambda(\tau)\left[r(\tau) a(\tau) + w(\tau) - e_r(\tau)\right] \\ &= \ln E(\tau) - \ln L - \ln P(\tau)^{\frac{1}{1-\varepsilon}} + \lambda(\tau)\left[r(\tau) a(\tau) + w(\tau) - \frac{E(\tau)}{L}\right]\end{aligned}$$

  - We obtain the Keynes-Ramsey rule for optimal expenditure (see Exercise 6.1.8 for derivation)

$$\frac{\dot{E}(\tau)}{E(\tau)} = r(\tau) - \rho, \tag{4.10}$$

    meaning that total consumption expenditure rises if the interest rate exceeds the time preference rate

- Intermediate good firms
  - Each firm has its own variety and can therefore act as a monopolist ...
  - ... subject to competition from other firms that offer their varieties (model of "monopolistic competition")
  - Price charged by firm i (see Exercise 6.1.9)

    $$p\left(i\right)=\frac{w}{\theta}$$

    where w represents marginal costs from production (compare the production function) and $1/\theta > 1$ is the mark-up over marginal costs
  - Pricing equation implies a symmetric equilibrium

    $$p\left(i\right)=p$$

  - All firms charge the same price p

- R&D firms
  - Research is undertaken ($L_R > 0$) as long as

$$v \geq \frac{w}{\varphi}$$

    i.e. as long as payoff from R&D v exceeds (or just equals) the costs $\frac{w}{\varphi}$
  - This equation comes from profit maximization of R&D firms OR we think of free market entry condition
  - In equilibrium (with ongoing innovation)

$$v = \frac{w}{\varphi}$$

4.5 Equilibrium without choosing a numeraire

- To summarise, the three equations describing knowledge accumulation, firm value growth and expenditure growth are (see Exercise 6.1.10 for derivation)

$$\dot{n}(t) = \varphi L - \theta \frac{E(t)}{v(t)},$$
$$\frac{\dot{v}(t)}{v(t)} = r(t) - (1-\theta)\frac{E(t)}{v(t)\,n(t)},$$
$$\frac{\dot{E}(t)}{E(t)} = r(t) - \rho.$$

- Let us define $x(t) \equiv E(t)/v(t)$. Then we get

$$\frac{\dot{x}(t)}{x(t)} = \frac{\dot{E}(t)}{E(t)} - \frac{\dot{v}(t)}{v(t)}$$

- Why should we do this? Neither $E(t)$ nor $v(t)$ appear independently

- Using the last two equations above gives a system of two differential equations

$$\dot{n}(t) = \varphi L - x(t)\,\theta$$
$$\frac{\dot{x}(t)}{x(t)} = (1-\theta)\frac{x(t)}{n(t)} - \rho$$

- This completes our derivation of a reduced form without choosing a numeraire, i.e. without normalising a price to one
- This is very useful as a consistency check for the structure of the model and the rearrangements

4.6 Phase diagram illustration

- For comparability reasons with literature, we now go back to a system in $v(t)$ and $n(t)$ by setting $E(t) = 1$ (any nominal quantity would do)

- The reduced form reads

$$\frac{\dot{n}(t)}{n(t)} = \frac{\varphi L}{n(t)} - \frac{\theta}{n(t)\,v(t)}$$
$$\frac{\dot{v}(t)}{v(t)} = \rho - \frac{1-\theta}{n(t)\,v(t)}$$

- Equilibrium analysis shows (see Exercise 6.1.10) that

  - there is temporary innovation but
  - no long-run growth
  - the number of varieties increases up to some maximum level after which
  - the economy comes to a halt – growth peters out
  - see phase diagram on the next slide

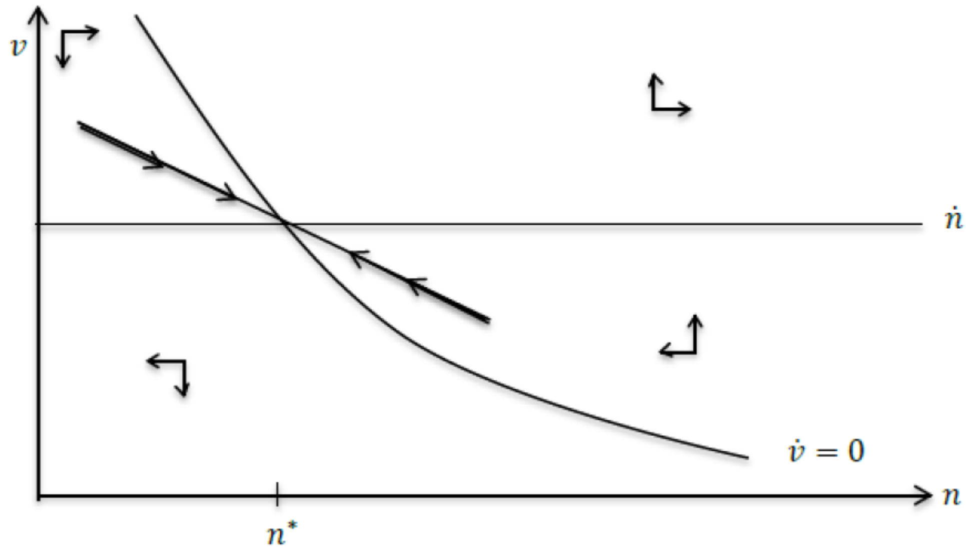


Figure 13 *Phase diagram for the model with innovation (and without knowledge spillovers)*

4.7 Knowledge spillovers yield long-run growth

- Idea
  - Researchers stand "on giants' shoulders"
  - Doing R&D does not only lead to a new variety, but it also creates knowledge
  - This knowledge $K(n(t))$ is a public good and available for others afterwards

$$\dot{n}(t) = \varphi L_R(t) K(n(t)), \tag{4.11}$$
$$K(n(t)) = n(t)$$

  - (Compare to earlier R&D equation in (4.1))

- Reduced form
  - Again, we obtain a system in number of varieties and value of a representative variety

$$\frac{\dot{n}(t)}{n(t)} = \varphi L - \frac{\theta}{n(t)\,v(t)}$$
$$\frac{\dot{v}(t)}{v(t)} = \rho - \frac{1-\theta}{n(t)\,v(t)}$$

- Long-run growth
  - First question (as always): is there some type of steady state or balanced growth path?
  - We guess that there is a constant growth rate g with $\dot{n}(t)/n(t) = -\dot{v}(t)/v(t) = g$
  - Verify that this makes sense and compute g by plugging guess into reduced form

$$g = \varphi L - \frac{\theta}{n(t)\,v(t)}$$
$$-g = \rho - \frac{1-\theta}{n(t)\,v(t)}$$

  - Next question: what is g and what is $n(t)\,v(t)$ (which are constant on balanced growth path)?
  - Solving this two-equation system for g and $n(t)\,v(t)$ gives our endogenous growth rate in this economy

$$g = (1-\theta)\,\varphi L - \theta\rho \tag{4.12}$$

- Determinants of endogenous growth rate
  - Growth rate depends on economic determinants – in (very strong) contrast to Solow growth model where g is an exogenous parameter
  - The higher productivity of R&D workers (φ), the larger the economy (L), the more patient individuals (lower ρ) and the lower the elasticity of substitution (lower θ), the higher the growth rate of the economy
  - Low θ means high markups and profits for firms, i.e. high incentives to do R&D
  - Growth could also be zero, there would be no growth
  - This is a "scale-economy" – the larger (L) the economy, the faster it grows

- What if the economy is not on the balanced growth path?
  - Look at a phase diagram (see next slide)
  - Construct zero-motion lines (and suppress time arguments)

$$\frac{\dot{n}}{n} \geq 0 \Leftrightarrow \varphi L \geq \frac{\theta}{nv} \Leftrightarrow v \geq \frac{\theta}{n\varphi L}$$
$$\frac{\dot{v}}{v} \geq 0 \Leftrightarrow \rho \geq \frac{1-\theta}{nv} \Leftrightarrow v \geq \frac{1-\theta}{n\rho}$$

  - Ask, where in phase diagram n rises and v falls
  - Do the $\frac{\dot{n}}{n} = 0$ loci lie below the $\frac{\dot{v}}{v} = 0$ loci (as drawn)?
    * Yes, for a parameter condition
    * If the growth rate from (4.12) is positive, i.e. if $g > 0$
  - After having plotted pairs of arrows, we see that there are three (types of) trajectories for a given initial level n_0 of varieties
  - Which path is the only reasonable one? The one that lies on the balanced growth path
  - This gives unique v_0, i.e. the unique initial value for a representative firm

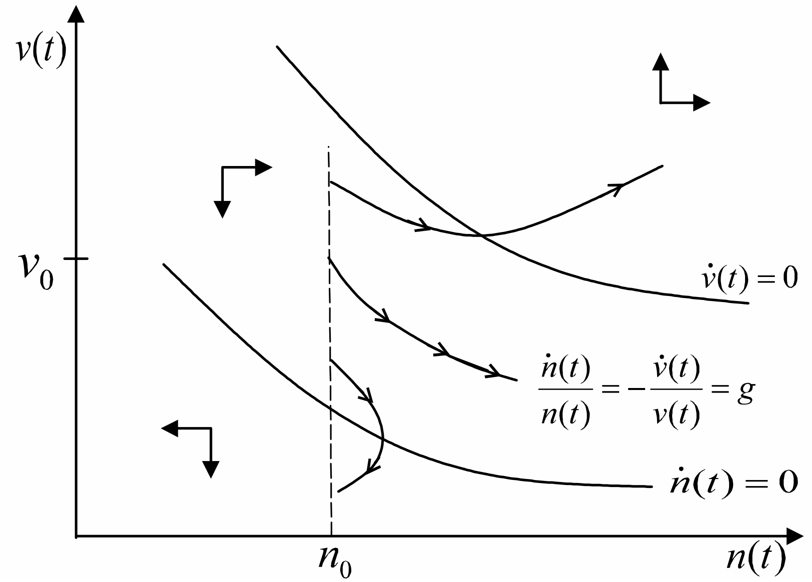


Figure 14 *Phase diagram for the model with innovation (and with knowledge spillovers)*

4.8 Non-scale models

(not covered in lecture – *sine qua non* for good research on growth, not important for exam)

4.8.1 The empirical background

- Jones (1995a,b) argued that scale-dependence of growth rate contradicts data

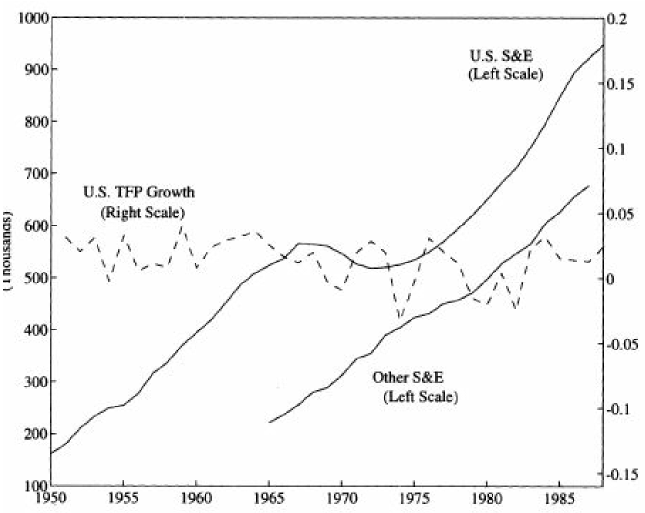


Figure 15 *Scientists and engineers engaged in R&D and U.S. TFP growth. "Other S&E" is the sum of scientists and engineers engaged in R&D for France, Germany, and Japan. Source: Jones (1995a, fig. 1)*

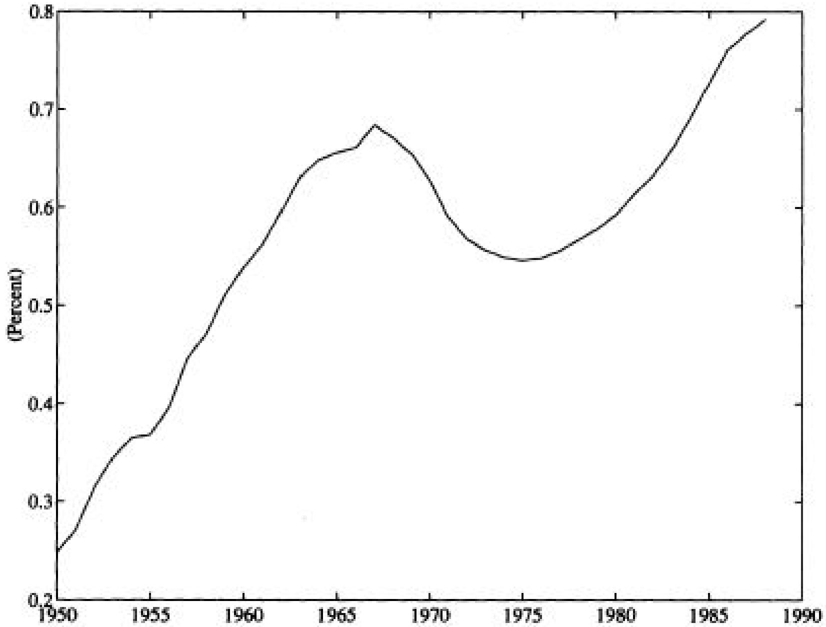


Figure 16 *U.S scientists and engineers engaged in R&D as a share of total labour force. Source: Jones (1995a, fig. 2)*

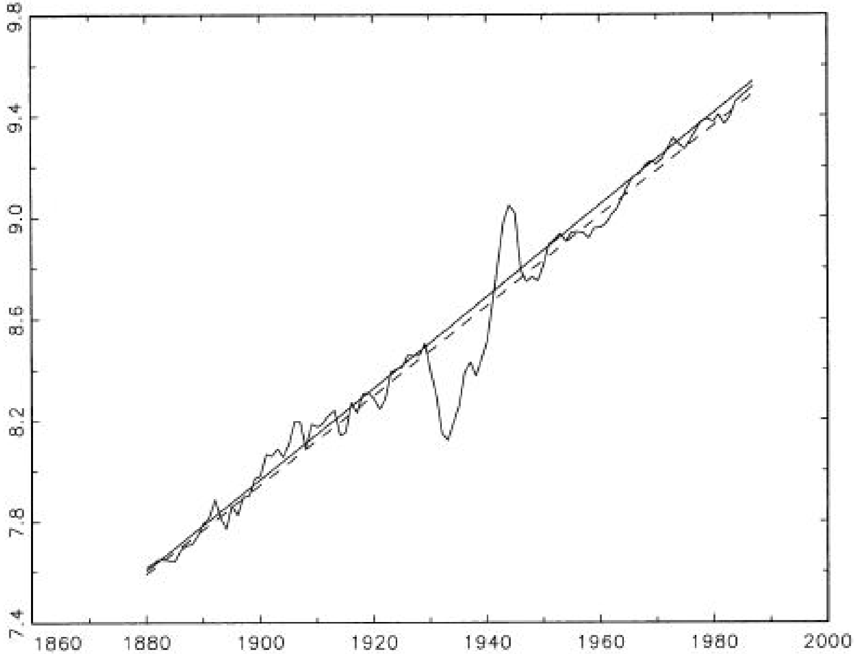


Figure 17 *Per capita GDP in the US, 1880-1987 (Natural logarithm). The solid trend line represents the time trend calculated using data only from 1880 to 1929. The dashed line is the trend for the entire sample. Source: Jones (1995b, fig. 1)*

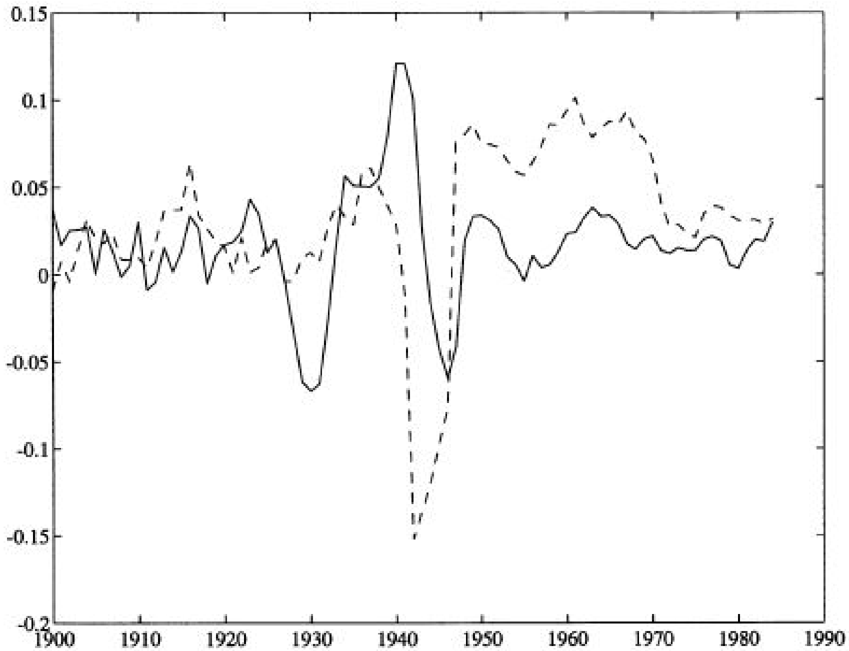


Figure 18 *Annual growth rates for the US (solid) and Japan (dashed), 1900-1987. Source: Jones (1995b, fig. 2)*

- Reading of these figures
  - The number of researchers and scientists goes up
  - The growth rate of the economy (US) remains the same
  - If R&D equation (4.11) was correct ...

$$\frac{\dot{n}(t)}{n(t)} = \varphi L_R(t)$$

    ... an increase in number of researchers (L_R) would imply an increase in growth rate (of varieties and output)
  - New model structure needed

4.8.2 The theoretical explanation

- Jones (1999) summarizes how non-scale models work
- These models are semi-endogenous growth models as the growth rate of TFP or GDP per capita is positive only with a positive population growth rate
- The specification of the R&D process is $\dot{A}(t) = \delta L_A(t) A(t)^{\phi}$ or (in notation as in (4.11))

$$\dot{n}(t) = \varphi L_R(t) K(n(t))$$
$$K(n(t)) = n(t)^{\phi}$$

- What is the value of ϕ?
  - $\phi = 1$ is the specification of the Grossman Helpman model (or of Romer, 1990, Aghion and Howitt, 1992 and many others) in (4.11)
  - $0 < \phi < 1$ allows for positive externalities but at decreasing returns (which is assumed)
  - $\phi = 0$ implies the absence of externalities – as in (4.1)
  - $\phi < 1$ captures the idea that finding new ideas is more difficult when many ideas are already around

- The growth rate
  - With this specification the growth rate of TFP on the balanced growth path is

    $$g_A = \frac{n}{1-\phi}$$

    where n is the growth rate of the population
  - The growth rate is positive obviously only for positive population growth
  - This is NOT an exogenous growth model a la Solow, however, as there are endogenous investment decisions of firms
  - It is a semi-endogenous growth model, however

4.9 What have we learned?

- Why do countries grow?
  - Number of varieties grows because of positive knowledge externalities in the R&D process
  - Consumption per capita and GDP and GDP per capita grow for the same reason

- Why are some countries richer than others (in terms of GDP per capita)?
  - Catching up does not necessarily take place (strong difference to Solow model)
  - All countries jump immediately (compare figure 14) on the balanced growth path
  - All countries always grow with g (which is the same as long as parameters that determine g are the same)
  - relative income differences can persist forever

- Why do some countries grow faster than others?
  - they do not, as long as g is the same across countries
- Can countries grow temporarily faster or also permanently?
  - There is conditional catching-up, overtaking and falling behind (as there is conditional convergence in Solow model)
  - "Conditional" means "having same parameters and policies"
  - If countries differ in their parameters, they can grow differently

- Is the growth rate optimal?
  - General principle: growth rate might not be optimal due to some inefficiency/ market-failure
  - Growth rate can be too high or too low
  - Growth rate here might not be optimal due to knowledge externality (one example of market failure)
  - In fact, it is too low because the externality is a positive externality
  - The growth rate could also be too high (for different externalities, see below)
- New growth theory offers fundamentally different (and much richer) views on economic growth than the Solow growth model

5 New growth theory: Major innovations

5.1 The questions

- How does a growth process look like?
- Is it just better technologies where everybody benefits?
  - Think of Solow model with "labour-saving" technological progress
  - Think of reallocation of labour to new activities as in endogenous growth models with new varieties
- Or is there some sort of Schumpeterian "creative destruction" going on?
  - Creative destruction means that the growth process comes from "new methods of production, ..., new markets, new forms of industrial organization ... that capitalist enterprise creates" (Schumpeter, Capitalism, Socialism and Democracy, p. 82 ff)
  - "industrial mutation ... revolutionizes the economic structure from within, incessantly destroying the old one"

- References
  - Aghion and Howitt (1992) “A Model Of Growth Through Creative Destruction“
  - Aghion and Howitt (1994) “Growth and Unemployment”
  - Wälde (1999) replaces risk-neutral by risk-averse households
  - much more subsequent work

5.2 The production side

- First sector (out of three) produces the consumption good y
  - The firm uses the technology
    $$y(t) = \gamma^t x(t)^\alpha h^{1-\alpha} \equiv \gamma^t x(t)^\alpha$$
    by employing an intermediate good $x(t)$ and some indivisible factor h (the entrepreneur) which we normalize to one
  - The technological level is given by γ^t where $\gamma > 1$ and t is the currently most advanced technology (t is not time, this is Aghion-Howitt's original notation)
  - The technology comes with the intermediate good x (it is embodied in x)
  - Firms are price takers and maximize profits. Profits amount to (see Exercise 6.1.11)
    $$\pi_y(t) = (1-\alpha)\gamma^t x(t)^\alpha \tag{5.1}$$
  - Interpretation for perfect competition: $\pi_y(t)$ are factor rewards to h

- The second sector produces the intermediate good
  - This is the "Schumpetarian sector" where creative destruction takes place
  - Intermediate good x is produced by a monopolist employing labour L

$$x(t) = L(t)$$

  - Optimal price is a markup $1/\alpha$ over marginal cost w (see Exercise 6.1.9)

$$p(t) = \frac{w(t)}{\alpha}$$

    where $1/(1-\alpha)$ is the price elasticity of demand resulting from the consumption good sector
  - Profits of the monopolist amount to

$$\pi_x(t) = \alpha \pi_y(t) \tag{5.2}$$

- The third sector undertakes R&D
  - Research is risky and does not necessarily lead to a successful end
  - Big difference to deterministic growth models where research can be planned and predicted perfectly
  - Riskiness is modelled by a Poisson process

5.3 Excursion on Poisson processes [background]

5.3.1 What are stochastic processes? [background]

- In some loose sense, a random variable relates to a stochastic process as (deterministic) static models relate to (deterministic) dynamic models
  - Static models describe one equilibrium, dynamic models describe a sequence of equilibria
  - A random variable has, "when looked at once" (e.g. when throwing a die once), one realization. A stochastic process describes a sequence of random variables and therefore, "when looked at once", describes a sequence of realizations.

Definition 2 *(Ross, 1996) A stochastic process is a parameterized collection of random variables,* $\{X(t)\}_{t\in[t_0,T]}$.

- Stochastic processes can be
  - stationary
  - weakly stationary or
  - non-stationary
- Stationarity is a more restrictive concept than weak stationarity

Definition 3 *(Ross, 1996, ch. 8.8): A process $X(t)$ is stationary if $X(t_1), ..., X(t_n)$ and $X(t_1+s), ..., X(t_n+s)$ have the same joint distribution for all n and s.*

- An implication of this definition, which might help to get some "feeling" for this definition, is that a stationary process $X(t)$ implies that, being in $t=0$, $X(t_1)$ and $X(t_2)$ have the same distribution for all $t_2 > t_1 > 0$

Definition 4 *(Ross, 1996) A process $X(t)$ is weakly stationary if the first two moments are the same for all t and the covariance between $X(t_2)$ and $X(t_1)$ depends only on $t_2 - t_1$,*

$$E_0 X(t) = \mu, \quad Var X(t) = \sigma^2, \quad Cov(X(t_2), X(t_1)) = f(t_2 - t_1),$$

where μ and σ^2 are constants and $f(.)$ is some function.

- A process which is neither stationary nor weakly stationary is non-stationary.
- What is the (probably) best-known stochastic process in continuous time?
  - Brownian motion
  - Sometimes called the Wiener process after a mathematician Wiener who provided the following definition

Definition 5 *(Ross, 1996) Brownian motion*
A stochastic process $z(t)$ is a Brownian motion process if (i) $z(0) = 0$, (ii) the process has stationary independent increments and (iii) for every $t > 0$, $z(t)$ is normally distributed with mean 0 and variance $\sigma^2 t$.

- The first condition $z(0) = 0$ is a normalization (define $z(0) \equiv y(0) - y_0$)
- The second condition makes two statements about increments
  - Think of points in time $t_4 > t_3 \geq t_2 > t_1$. Then increments are $z(t_4) - z(t_3)$ or $z(t_2) - z(t_1)$ (as one of many examples)
  - Increments are random variables and they are *independent* of previous increments, i.e. $z(t_4) - z(t_3)$ independent of $z(t_2) - z(t_1)$
  - Formally and by definition of independence, joint distribution of $z(t_4) - z(t_3)$ and $z(t_2) - z(t_1)$ is given by product of individual distribution,

$$F(z(t_4) - z(t_3), z(t_2) - z(t_1)) = F_{34}(z(t_4) - z(t_3)) F_{12}(z(t_2) - z(t_1))$$

  - Increments are *stationary* if (see the above definition) the stochastic process $X(t) \equiv z(t) - z(t-s)$ has the same distribution for any t and constant s
- Finally, the third condition is the heart of the definition

  - $z(t)$ is normally distributed
  - The variance increases linearly in time
  - the Wiener process is therefore non-stationary

- Let us now define a stochastic process which plays also a major role in economics

Definition 6 *Poisson process (adapted following Ross 1993, p. 210)*
A stochastic process $q(t)$ *is a Poisson process with arrival rate* λ *if (i)* $q(0) = 0$*, (ii) the process has independent increments and (iii) the increment* $q(\tau) - q(t)$ *in any interval of length* $\tau - t$ *(the number of "jumps") is Poisson distributed with mean* $\lambda[\tau - t]$*, i.e.* $q(\tau) - q(t) \sim Poisson(\lambda[\tau - t])$.

- A Poisson process (and other related processes) are also sometimes called "counting processes" as $q(t)$ counts how often a jump has occurred, i.e. how often something has happened

- There is a close similarity in the first two points of this definition with the definition of Brownian motion

- The third point here means that the probability that the process increases n times between t and $\tau > t$ is given by

$$P[q(\tau) - q(t) = n] = e^{-\lambda[\tau - t]} \frac{(\lambda[\tau - t])^n}{n!}, \qquad n = 0, 1, ... \tag{5.3}$$

  This implies that one could think of as many stochastic processes as there are distributions, defining each process by the distribution of its increments.

5.3.2 An intuitive understanding of a Poisson process [background]

- What is a Poisson process? (see Wälde, 2012, Definition 10.1.6 for details)
  - A Poisson process $q(t)$ is a stochastic process in continuous time
  - A stochastic process is a collection of random variables $q(t)$ at points in time t
  - A Poisson process can be characterized by its increment over a short period of time dt

    $$dq(t) = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right\} \text{ with probability } \left\{ \begin{array}{c} \mu \text{dt} \\ 1 - \mu \text{dt} \end{array} \right\}$$

    where μ is the arrival rate
  - Why called Poisson process? The number of jumps between t and $\tau > t$ is Poisson-distributed with parameter $\mu [\tau - t]$

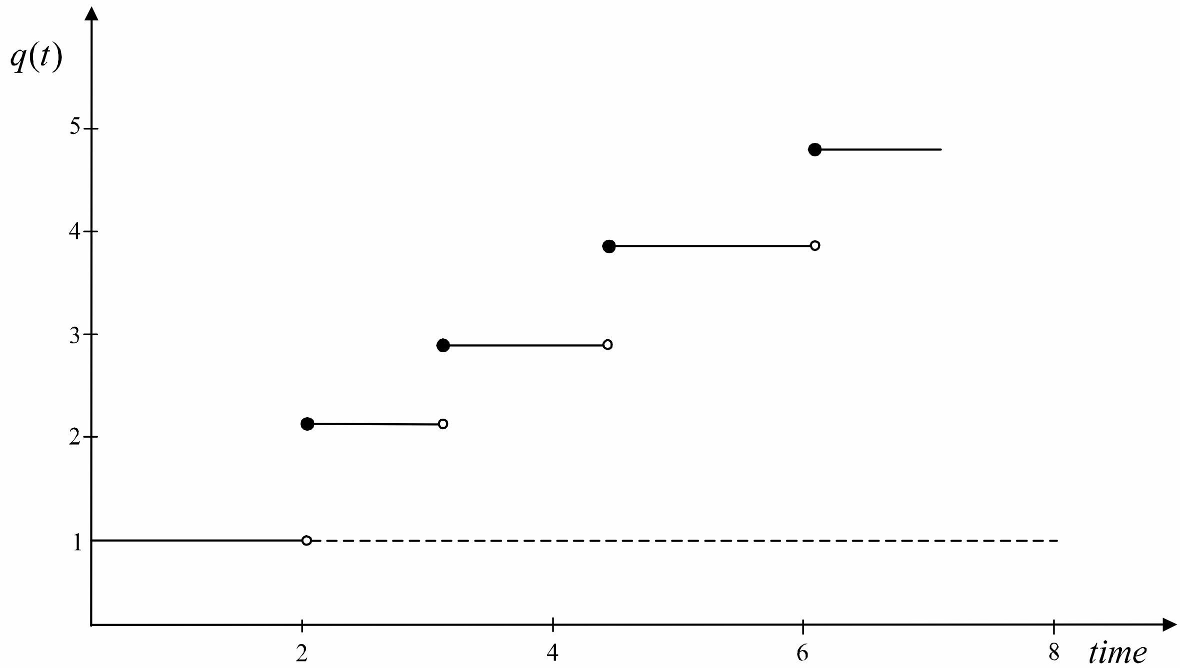


Figure 19 *Illustration of Poisson process (also called counting process as it counts the number of jumps* $dq(t)$*)*

- [back to slide 5.3] The third sector undertakes R&D
  - The arrival rate for success in R&D is

$$\Gamma(t) = \lambda n(t) \tag{5.4}$$

    where λ is a parameter and $n(t)$ is the number of engineers (workers) employed in this sector
  - (see similarity to R&D equation (4.11) in model by Grossman and Helpman)
  - When a firm has success in R&D, it knows how to produce an intermediate good that implies a productivity of γ^{t+1}
  - Firm drives intermediate monopolist (see above) out of market (Schumpeterian creative destruction) and earns profits $\pi_x(t)$
  - These profits provide incentives to do R&D

- How is R&D financed? (Wälde, 1999)
  - R&D firms sell shares $\chi(t)$ to households at a price $\omega(t)$
  - Selling the share commits the firm to hire φ workers for research
  - Letting R&D firms act under perfect competition, total receipts $\omega(t)\chi(t)$ from selling shares equal R&D costs $w(t)n(t)$

$$\omega(t)\chi(t) = w(t)n(t) \tag{5.5}$$

  - When R&D fails, shares are worthless
  - When R&D is successful, households own the new intermediate monopolists
  - As the latter makes profits in intermediate goods market, households are (in expectations, i.e. on average) rewarded for their buying of shares by these profits
  - The only way households can invest in this economy is by buying shares in R&D projects. Aggregate investment $I(t)$ therefore equals total receipts of R&D firms from selling their shares. This implies that, using (5.5),

$$I(t) = w(t)n(t).$$

5.4 Labour market

- Labour is the only factor that is mobile between sectors
- Total supply is N
- Wage rate $w(t)$ is determined on the labour market which is assumed to clear at every moment in time

$$n(t) + L(t) = N$$

5.5 Consumers

5.5.1 Preferences and constraints

- Households maximize an intertemporal utility function (as in Cass-Koopmans-Ramsey model)

$$U(t) = \int_t^{\infty} e^{-\rho[\tau - t]} u\left(c(\tau)\right) d\tau, \tag{5.6}$$

where instantaneous utility is given by

$$u(c) = c^{\sigma}. \tag{5.7}$$

- These preferences imply an intertemporal elasticity of substitution of $\varepsilon = (1-\sigma)^{-1}$ and a constant relative risk aversion of ε^{-1}.

- The intratemporal budget constraint is given by

$$c(t) + i(t) = w(t) + \pi(t) \tag{5.8}$$

saying that consumption expenditure $c(t)$ plus investment $i(t)$ equals labour income $w(t)$ plus capital income $\pi(t)$

- Capital income $\pi(t)$ results from firms active in the consumption good sector (5.1) and from shares $s(t)$ held in the intermediate good sector (5.2),

$$\pi(t) = \pi(s(t)) = N^{-1}\pi_y(t) + s(t)\,\pi_x(t)$$

- The dynamic budget constraint
  - A household receives a share $s^{\text{new}}(t)$ of total profits that is given by her investment $i(t)$ relative to total investment $I(t)$ made into the successful research project,

$$s^{\text{new}}(t) = \frac{i(t)}{I(t)} \leq 1$$

  - A successful research project also implies that the old monopolist is driven out of the market and that all shares held by a household in the old monopolist lose their value
  - When research projects are not successful, the amount of shares owned by the household does not change
  - All of this is captured by the following stochastic differential equation

$$ds(t) = \left(\frac{i(t)}{I(t)} - s(t)\right) dq(t), \tag{5.9}$$

  where $q(t)$ is the R&D Poisson process with the arrival rate $\Gamma(t)$ from (5.4)

5.5.2 Stochastic differential equations (SDEs) [background]

- To understand the above (and related) structures a bit better, let us look at SDEs more generally
- This follows Wälde (2012, ch.10)
- Sources of uncertainty in SDEs include
  - Brownian motions (frequently used e.g. for asset price modelling)
  - Poisson process (R&D, search & matching models, finance, international macro ...)
  - Levy processes (more advanced, mainly in mathematical finance literature)

- Brownian motion with drift
$$dx(t) = adt + bdz(t). \tag{5.10}$$
  - Assume increments have a standard normal distribution, i.e. $E_t[z(\tau) - z(t)] = 0$ and $\text{var}_t[z(\tau) - z(t)] = \tau - t$. We will call this standard Brownian motion
  - Constant a frequently called drift rate, b^2 is sometimes referred to as the variance rate of $x(t)$
    * The expected increase of $x(t)$ is determined by a only (and not by b)
    * The variance of $x(\tau)$ for some future $\tau > t$ is only determined by b
  - The drift rate a is multiplied by dt, a "short" time interval, the variance parameter b is multiplied by $dz(t)$, the increment of the Brownian motion process $z(t)$ over a small time interval
  - We can interpret (5.10) by comparing with the following simple ordinary differential equation
$$\dot{y}(t) = a \tag{5.11}$$
  whose solution is $y(t) = y_0 + at$. When we draw this solution and also the above SDE for three different realizations of $z(t)$, we obtain the following figure

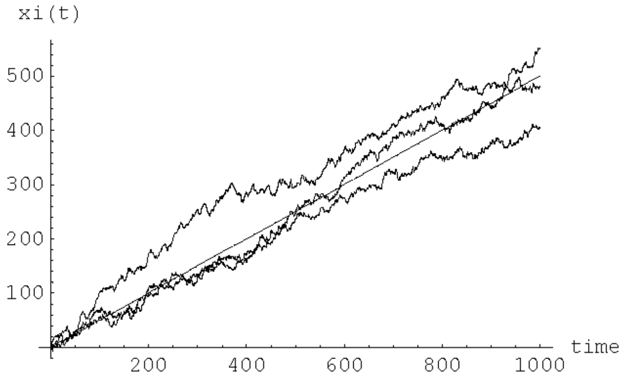


Figure 20 *The solution of the deterministic differential equation (5.11) and three realizations of the related stochastic differential equation (5.10)*

– Adding a stochastic component to the differential equation leads to fluctuations around the deterministic path

– Clearly, how much the solution of the SDE differs from the deterministic one is random, i.e. unknown

– We will understand later that the solution of the deterministic differential equation (5.11) is identical to the evolution of the expected value of $x(t)$, i.e. $y(t) = E_0 x(t)$ for $t > 0$

- Generalized Brownian motions (Ito processes)

$$dx(t) = a(x(t), z(t), t)\, dt + b(x(t), z(t), t)\, dz(t) \tag{5.12}$$

  - One can also refer to $a(.)$ as the drift rate and to $b^2(.)$ as the instantaneous variance rate
  - These functions can be stochastic themselves
  - In addition to arguments $x(t)$ and time, Brownian motion $z(t)$ can be included in these arguments
  - Thinking of (5.12) as a budget constraint of a household, an example could be that wage income or the interest rate depend on the current realization of the economy's fundamental source of uncertainty, which is $z(t)$

- Stochastic differential equations with Poisson processes

$$dx(t) = adt + bdq(t) \tag{5.13}$$

  - (Slightly more general than budget constraint in (5.9))
  - As long as no jump occurs, i.e. as long as $dq = 0$, the variable $x(t)$ follows $dx(t) = adt$ which means linear growth, $x(t) = x_0 + at$
  - When q jumps, i.e. $dq = 1$, $x(t)$ increases by b
    * Write $dx(t) = \tilde{x}(t) - x(t)$, where $\tilde{x}(t)$ is the level of x immediately after the jump
    * Let the jump be "very fast" such that $dt = 0$ during the jump
    * Hence, $\tilde{x}(t) - x(t) = b \cdot 1$, where the 1 stems from $dq(t) = 1$, or

$$\tilde{x}(t) = x(t) + b \tag{5.14}$$

  - Clearly, the points in time when a jump occurs are random. A tilde ($\tilde{}$) will always denote in what (and in various papers in the literature) follows the value of a quantity immediately after a jump.

- A realization of the process described in (5.13) for $x(0) = x_0$ is depicted in Figure (22)

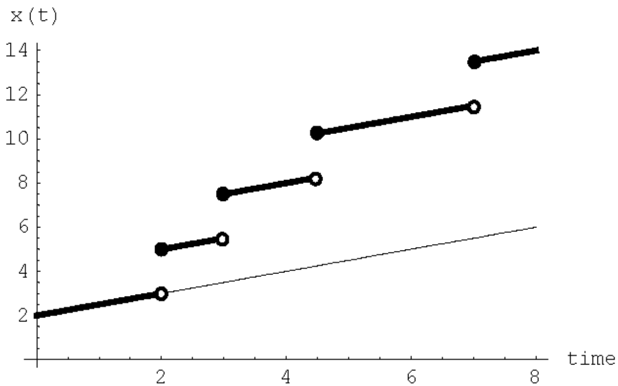


Figure 21 ***Figure 22*** *An example of a Poisson process with drift (thick line) and a deterministic differential equation (thin line)*

- Poisson process vs. Brownian motion
  - In contrast to Brownian motion, a Poisson process contributes to the increase of the variable of interest
    - Without the $dq(t)$ term (i.e. for $b = 0$), $x(t)$ would follow the thin line
    - With occasional jumps, $x(t)$ grows faster
  - In the Brownian motion case of the figure before, realizations of $x(t)$ remained "close to" the deterministic solution
  - This is simply due to the fact that
    - the expected increment of Brownian motion is zero while
    - the expected increment of a Poisson process is positive

- A geometric Poisson process

$$dx(t) = a(q(t), t)\, x(t)\, dt + b(q(t), t)\, x(t)\, dq(t) \tag{5.15}$$

  - Processes are usually called geometric when they describe the rate of change of some RV $x(t)$, i.e. $dx(t)/x(t)$ is not a function of $x(t)$
  - The deterministic part shows that $x(t)$ grows at the rate of $a(.)$ in a deterministic way and jumps by $b(.)$ percent, when $q(t)$ jumps
  - In contrast to a Brownian motion SDE, $a(.)$ here is *not* the average growth rate of $x(t)$
  - Geometric Poisson processes can be used to describe the evolution of asset prices in a simple way
    * There is some deterministic growth component $a(.)$ and some stochastic component $b(.)$
    * When $b(.) > 0$, this could reflect new technologies in the economy
    * When $b(.) < 0$, this equation could be used to model negative shocks like oil-price shocks or natural disasters

- Aggregate uncertainty and random jumps
  - Extension of a Poisson differential equation makes the amplitude of the jump random
  - Starting from $dA(t) = bA(t)\, dq(t)$, where b is a constant, we can now assume that $b(t)$ is governed by some distribution, i.e.

$$dA(t) = b(t)\, A(t)\, dq(t), \qquad \text{where } b(t) \sim \left(\mu, \sigma^2\right) \tag{5.16}$$

  - Assume that $A(t)$ is total factor productivity in an economy. Then, $A(t)$ does not change as long as $dq(t) = 0$. When $q(t)$ jumps, $A(t)$ changes by $b(t)$, i.e.

$$dA(t) \equiv \tilde{A}(t) - A(t) = b(t)\, A(t) \Leftrightarrow$$

$$\tilde{A}(t) = (1 + b(t))\, A(t), \quad \forall t \text{ where } q(t) \text{ jumps}$$

  - This equation says that whenever a jump occurs, $A(t)$ increases by $b(t)$ percent, i.e. by the realization of the random variable $b(t)$
  - Obviously, the realization of $b(t)$ matters only for points in time where $q(t)$ jumps.
  - For an example, see the evolution of the states of the economy in the Pissarides-type matching model of Shimer (2005)

5.5.3 Differentials for stochastic differential equations [background]

- Functions of stochastic processes in continuous time need to be treated differently when it comes to “derivatives” or differentials
  - Brownian motion is not differentiable with respect to time (continuous with kinks)
  - Poisson process is not differentiable at jumps (discontinuity)
- Yet, differentials are needed all of the time in economics
- What should be done? We need to employ
  - (versions of) Ito’s Lemma (see Wälde, 2012, ch. 10.2.2)
  - Change-of-variable formulas for Poisson processes

- One stochastic process
  - Let there be a stochastic process $x(t)$ driven by Poisson uncertainty $q(t)$ described by

$$dx(t) = a(.)\,dt + b(.)\,dq(t)$$

  - Consider the function $F(t, x)$. The differential of this function is

$$dF(t,x) = F_t dt + F_x a(.)\,dt + \{F(t, x + b(.)) - F(t, x)\}\,dq \tag{5.17}$$

  - Recall that the functions $a(.)$ and $b(.)$ in the deterministic and stochastic part of this SDE can have as arguments any combinations of $q(t)$, $x(t)$ and t or can be simple constants
  - The rule in (5.17) is very intuitive: the differential of a function is given by the "normal terms" and by a "jump term"
    - The "normal terms" include the partial derivatives with respect to time t and x times changes per unit of time (1 for the first argument and $a(.)$ for x) times dt
    - Whenever the process q increases, x increases by the $b(.)$. The "jump term" therefore captures that the function $F(.)$ jumps from $F(t, x)$ to $F(t, \tilde{x}) = F(t, x + b(.))$.

- Two stochastic processes
  - Let there be two independent Poisson processes q_x and q_y driving two stochastic processes $x(t)$ and $y(t)$,

$$dx = a(.)\,dt + b(.)\,dq_x, \qquad dy = c(.)\,dt + g(.)\,dq_y$$

    and consider the function $F(x, y)$
  - The differential of this function is

$$\begin{aligned} dF(x,y) = \{F_x a(.) + F_y c(.)\}\,dt + \{F(x + b(.), y) - F(x,y)\}\,dq_x \\ + \{F(x, y + g(.)) - F(x,y)\}\,dq_y. \end{aligned} \tag{5.18}$$

  - This "differentiation rule" consists of the "normal" terms and the "jump terms"
  - As $F(.)$ has two arguments, the normal term contains two drift components, $F_x a(.)$ and $F_y c(.)$, and the jump term contains the effect of jumps in q_x and in q_y
  - The dt term does not contain time derivative $F_t(x, y)$ as $F(x, y)$ is not a function of time (extensions possible)
  - Basically, (5.18) is just the "sum" of two versions of (5.17). This is due to the fact that any two Poisson processes are, by construction, independent

- Economy-wide uncertainty
  - Consider now a economically frequent case
    - with one economy-wide source of uncertainty (new technologies, commodity price shocks) occuring according to some Poisson process
    - that affects many variables in this economy (e.g. all relative prices) simultaneously
  - When there are two variables x and y following

$$dx = a(.)\,dt + b(.)\,dq, \qquad dy = c(.)\,dt + g(.)\,dq,$$

    where uncertainty stems from the same q for both variables, the differential of a function $F(x, y)$ is

$$dF(x, y) = \{F_x a(.) + F_y c(.)\}\,dt + \{F(x + b(.), y + g(.)) - F(x, y)\}\,dq.$$

  - One nice feature about differentiation rules for Poisson processes is their very intuitive structure
    - With two *independent* Poisson processes as in (5.18), the change in F is given by either $F(x + b(.), y) - F(x, y)$ *or* by $F(x, y + g(.)) - F(x, y)$
    - When both arguments x and y are affected by the same Poisson process, the change in F is given by $F(x + b(.), y + g(.)) - F(x, y)$, i.e. the level of F after a simultaneous change of both x and y minus the pre-jump level $F(x, y)$

- Many stochastic processes [for information only]

  - We now present the most general case. Let there be n stochastic processes $x_i(t)$ and define the vector $x(t) = (x_1(t), ..., x_n(t))^T$. Let stochastic processes be described by n SDEs

$$dx_i(t) = \alpha_i(.)\,dt + \beta_{i1}(.)\,dq_1 + ... + \beta_{im}(.)\,dq_m, \quad i = 1, \ldots, n, \tag{5.19}$$

    where $\beta_{ij}(.)$ stands for $\beta_{ij}(t, x(t))$. Each stochastic process $x_i(t)$ is driven by the same m Poisson processes. The impact of Poisson process q_j on $x_i(t)$ is captured by $\beta_{ij}(.)$

Proposition 1 *Let there be n stochastic processes described by (5.19). For a once continuously differentiable function $F(t,x)$, the process $F(t,x)$ obeys*

$$dF(t,x(t)) = \{F_t(.) + \Sigma_{i=1}^n F_{x_i}(.)\,\alpha_i(.)\}\,dt + \Sigma_{j=1}^m \{F(t,x(t)+\beta_j(.)) - F(t,x(t))\}\,dq_j, \tag{5.20}$$

where F_t and F_{x_i}, $i = 1,\ldots,n$, denote the partial derivatives of f with respect to t and x_i, respectively, and β_j stands for the n-dimensional vector function $(\beta_{1j},\ldots,\beta_{nj})^T$

– The intuitive understanding is again simplified by focusing on "normal" continuous terms and on "jump terms". The continuous terms are as before and simply describe the impact of the $\alpha_i(.)$ in (5.19) on $F(.)$. The jump terms show how $F(.)$ changes from $F(t,x(t))$ to $F(t,x(t)+\beta_j(.))$ when Poisson process j jumps. The argument $x(t)+\beta_j(.)$ after the jump of q_j is obtained by adding β_{ij} to component x_i in x, i.e. $x(t)+\beta_j(.) = (x_1+\beta_{1j}, x_2+\beta_{2j}, ..., x_n+\beta_{nj})$.

5.6 Maximization problem

- Now that we understand
  - stochastic differential equations
  - differentials of functions of SDEs
- we can turn to
  - maximization problems with SDEs and
  - maximization problem for households in creative destruction framework

5.6.1 Bellman equations for Poisson processes [background]

- Imagine an individual that tries to maximize the objective function

$$U(t) = E_t \int_t^{\infty} e^{-\rho[\tau - t]} u(c(\tau)) \, d\tau \tag{5.21}$$

  subject to a budget constraint

$$da(t) = \{r(t) a(t) + w(t) - pc(t)\} dt + \beta a(t) dq(t) \tag{5.22}$$

- Defining the optimal program as $V(a) \equiv \max_{\{c(\tau)\}} U(t)$ subject to the constraint (5.22), Bellman equation is given by (see Wälde, 1999, Sennewald and Wälde, 2006, or Sennewald, 2007)

$$\rho V(a(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a(t)) \right\}. \tag{5.23}$$

- The Bellman equation has this basic form for "most" maximization problems in continuous time

- Starting point for other maximization problems as well (Poisson processes, Brownian motion or Levy processes)

- Given the general form of the Bellman equation in (5.23), we need to compute the differential $dV(a(t))$. Given the evolution of $a(t)$ in (5.22) and the CVF from (5.17), we find

$$dV(a) = V'(a)\{ra + w - pc\}dt + \{V(a + \beta a) - V(a)\}dq.$$

- In contrast to the CVF notation in for example (5.17), we now use simple derivative signs like $V'(a)$ as often as possible in contrast to for example $V_a(a)$. This is possible as long as functions have one argument only

- Forming expectations about $dV(a(t))$ gives

$$E_t dV(a(t)) = V'(a)\{ra + w - pc\}dt + \{V(\tilde{a}) - V(a)\}E_t dq.$$

  - The first term, the "dt-term" is known in t: The current state $a(t)$ and all prices are known and the shadow price $V'(a)$ is therefore also known
  - As a consequence, expectations need to be applied only to the "dq-term"
  - The first part of the "dq-term", the expression $V((1+\beta)a) - V(a)$ is also known in t as again $a(t)$, parameters and the function V are all non-stochastic
  - We therefore only have to compute expectations about dq. We know that $E_t[q(\tau) - q(t)] = \lambda[\tau - t]$. Now replace $q(\tau) - q(t)$ by dq and $\tau - t$ by dt and find $E_t dq = \lambda dt$

- The Bellman equation therefore reads

$$\rho V(a) = \max_{c(t)} \{u(c(t)) + V'(a)[ra + w - pc] + \lambda [V((1+\beta)a) - V(a)]\} \tag{5.24}$$

- The first-order condition is

$$u'(c) = V'(a)\, p \tag{5.25}$$

As always, (current) utility from an additional unit of consumption $u'(c)$ must equal (future) utility from an additional unit of wealth $V'(a)$, multiplied by the price p of the consumption good, i.e. by the number of units of wealth for which one can buy one unit of the consumption good

- The Keynes-Ramsey rule for the individual's problem reads (see Exercise 6.1.12)

$$-\frac{u''(c)}{u'(c)} dc = \left\{ r - \rho + \lambda \left[\frac{u'(\tilde{c})}{u'(c)} [1+\beta] - 1 \right] \right\} dt - \frac{u''(c)}{u'(c)} \{\tilde{c} - c\}\, dq. \tag{5.26}$$

  - The rule describes the evolution of consumption under optimal behaviour for a household that faces interest rate uncertainty resulting from Poisson processes
  - This equation is useful to understand, for example, economic fluctuations, wealth distributions and other

5.6.2 The maximization problem in the growth model

- We now return to our individual from slide 5.16
- The households maximises its utility (5.6), given instantaneous preferences (5.7) subject to (5.8) and (5.9) by choosing consumption. The Bellman equation reads

$$\rho V(s,t) = \max_{c} \left\{ u(c) + \Gamma \left[V\left(iI^{-1}, t+1\right) - V(s,t) \right] \right\} \tag{5.27}$$

- The first order condition is given by

$$u'(c) + \left[V(iI^{-1}, t+1) - V(s,t) \right] \frac{d}{dc}\Gamma + \Gamma \frac{d}{dc} \left[V(iI^{-1}, t+1) - V(s,t) \right] = 0 \tag{5.28}$$

- It says that marginal utility from consumption, the gain from the new technology $t+1$ times the marginal arrival rate and the expected marginal gain from the next technology must add up to zero
- See Wälde (1999) for more interpretation

- Solving by "guess and verify"
  - In various cases, the "guess and verify" approach is very useful
  - When it works, closed-form solutions are available (and not Keynes-Ramsey rules)
  - The idea consists in an "educated guess" and in proving that this guess satisfies all optimality conditions (see Wälde, 2011, for an overview)
- The guess here
  - We solve the problem by guessing that
    * optimal consumption is a share δ out of current income,

      $$c = \delta \left[w + \pi(s) \right],$$

      where δ does not depend on household specific variables like e.g. wage income and that
    * the value function is of the form

      $$V = \vartheta (w + \pi(s))^{\sigma},$$

      where ϑ is a constant

– Inserting these guesses into (5.27) and (5.28) allows to solve for δ

– Focusing on stationary equilibria and assuming the existence of a representative consumer, the consumption ratio δ is given by

$$\sigma\delta^{\sigma-1} = \frac{\delta^{\sigma}}{\rho - \Gamma(\gamma^{\sigma}-1)}\left[\frac{\alpha^2 N + (1-\alpha^2)L}{LN}\left(\gamma^2 - 1\right)\lambda\frac{L}{\alpha} + \Gamma\sigma\gamma^{\sigma}\frac{L}{n}\left(1 - N^{-1}\right)\frac{1-\alpha}{\alpha}\right]. \tag{5.29}$$

– This equation restates the first order condition (5.28) and says that consumption ratio δ is chosen such that marginal utility from current consumption on the left hand side is equal to discounted expected values of the gain from a new technology times the marginal arrival rate (the first term in brackets) plus the arrival rate times the marginal value from investment (the second term in the brackets)

5.7 Equilibrium

- Aggregate values
  - Aggregate consumption equals the number of households N times individual consumption and aggregate investment is given by

$$I = (1-\delta)(w+\pi)N \tag{5.30}$$

  - Using this aggregate investment equation and the fact that $I = wn$ allows to express labour demand in the R&D sector as a function of the individual consumption ratio δ. Inserting this demand function into the labour market clearing condition allows to express employment L in the intermediate good sector X as a function of δ. Solving for δ gives

$$\delta = \frac{L}{\alpha^2 N + (1-\alpha^2)L} \tag{5.31}$$

  - Equations (5.29) and (5.31) jointly determine δ and L
  - A unique equilibrium exists (see next slide)

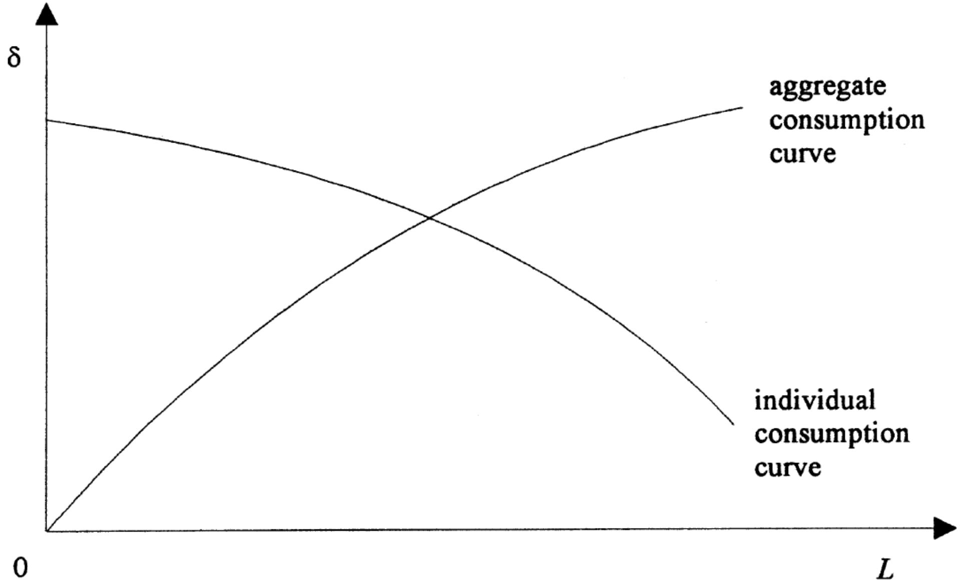


Figure 23 *Equilibrium consumption share δ and equilibrium production employment L (fig. 1, Wälde, 1999)*

- Schumpetarian equilibrium dynamics
  - There is a constant number n of researchers active in equilibrium (independent of technological level t)
  - The arrival rate for new technologies is constant and given by λn
  - The average waiting time between two innovations is $1/(\lambda n)$
  - With each new innovation, output increases by factor γ
  - All other variables also jump by this factor at random points in time
  - Equilibrium dynamics are close to trivial and look like a Poisson process in fig. 19

- Externalities (general)
  - Externalities are effects of consumption or production activity on agents other then the consumer or producer which do not work through the price system (based on J.J. Laffont, Externalities, New Palgrave Dict of Econ)
  - Examples: exhaust emissions of cars (negative), aircraft noise (negative), vaccination for a disease (positive), beekeeper helps pollination of crops (positive)
- Creative destruction and externalities
  - standing on giants shoulders: innovator does not take into account that his innovation increases output forever (too little investment)
  - Destructive effect of innovation, business stealing: innovator does not take into account that previous innovator is driven out of market (too much investment)
  - monopolistic behaviour: distortive price setting
  - and more, see Exercise 6.1.11, and Aghion and Howitt (1992)

5.8 What have we learned?

(in addition to the answers we heard above)

- What where our questions at the beginning of this part on growth?
  - Why do countries grow?
  - Why are some countries richer than others?
  - Why do some countries grow faster than others?
  - Can they grow temporarily faster or also permanently?
- One should make up one's mind whether one believes in
  - smooth and deterministic growth or
  - growth with turnovers, creative destruction and other disruptions

- Methods to deal with uncertainty in the growth prospect
  - Stochastic resource constraints
  - Stochastic budget constraints
  - Tools to solve maximization problems in the presence of uncertainty
  - Here, specifically: Brownian motion, Poisson processes, rules for computing differentials, stochastic Bellman equations and the like
- Implications for public policy questions
  - Is the growth rate optimal?
  - further externalities as just discussed show that there can be too much or too little growth
  - growth is not necessarily a very "peaceful process" where TFP rises smoothly (Solow) or more varieties are added to existing ones (Grossman and Helpman)
  - growth can be rather destructive (in the spirit of Schumpeter)

6 Exercises on economic growth

6.1 Exercises

6.1.1 Optimal Consumption

1. Consider the planner's objective function and dynamic budget constraint,

$$U = \int_t^{\infty} e^{-\rho(\tau - t)} u\left(C\left(\tau\right)\right) d\tau,$$
$$\dot{K}\left(t\right) = Y\left(K\left(t\right), L\right) - \delta K\left(t\right) - C\left(t\right),$$

where

$$u\left(C\left(t\right)\right) = \frac{C\left(t\right)^{1-\sigma} - 1}{1 - \sigma}, \sigma > 0.$$

   (a) Provide an interpretation of these equations.

   (b) Compute the Keynes-Ramsey rule for the planner's problem.

2. Consider the following objective function,

$$U = \int_t^{\infty} e^{-\rho(\tau - t)} u\left(c\left(\tau\right)\right) d\tau,$$

and dynamic budget constraint

$$\dot{a}(\tau) = r(\tau)a(\tau) + w(\tau) - c(\tau).$$

(a) Provide an interpretation to these equations.

(b) Compute the Keynes-Ramsey rule for the individual's savings problem.

(c) What is the Keynes-Ramsey rule if the instantaneous utility function takes a CRRA form

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

6.1.2 Properties of the CRRA utility function (background only)

1. Intertemporal elasticity of substitution – Why is intertemporal elasticity of substitution constant in: $\frac{c^{1-\sigma}-1}{1-\sigma}$?

2. Logarithmic utility function – What is the limit of $\frac{c^{1-\sigma}-1}{1-\sigma}$ as σ tends to 1? Use l'Hôpital's rule plot the function qualitatively for increasing levels of σ.

   Reminder: l'Hôpital's rule says that if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

6.1.3 Basics of dynamic programming

Using the objective function

$$U(t) = \int_t^\infty e^{-\rho[\tau-t]} u(c(\tau))\, d\tau$$

and the budget constraint

$$\dot{a}(t) = r(t)\, a(t) + w(t) - p(t)\, c(t),$$

derive the Keynes-Ramsey rule using dynamic programming.

6.1.4 Money in the utility function (background only)

Consider an individual with the following utility function

$$U(t) = \int_t^\infty e^{-\rho[\tau-t]} \left[\ln c(\tau) + \gamma \ln \frac{m(\tau)}{p(\tau)} \right] d\tau.$$

As always, ρ is the time preference rate and $c(\tau)$ is consumption. This utility function also captures demand for money by including a real monetary stock of $m(\tau)/p(\tau)$ in the utility function where $m(\tau)$ is the amount of cash and $p(\tau)$ is the price level of the economy. Let the budget constraint of the individual be

$$\dot{a}(t) = i(t)[a(t) - m(t)] + w(t) - T/L - p(t)\, c(t).$$

where $a(t)$ is the total wealth consisting of shares in firms plus money, $a(t) = k(t) + m(t)$ and $i(t)$ is the nominal interest rate.

1. Derive the budget constraint by assuming interest payments of i on shares in firms and zero interest rates on money.

2. Derive the optimal money demand.

6.1.5 Phase diagrams: a general introduction

1. General example

   A first-order autonomous differential equation is given as

   $$\dot{x}(t) = F(x(t)).$$

   (a) What is an equilibrium state of this equation? Provide an interpretation of the equilibrium state using the following general expression

   $$\dot{x}(t) = Ax(t)^3 + Bx(t)^2 - Cx(t),$$

   with $A, B, C > 0$. Draw the corresponding phase diagram, and discuss stable and unstable equilibria.

(b) What are the properties of these equilibria?

(c) Draw the phase diagram for the Lotka-Volterra model (predator-prey model) given by the differential equation system

$$\dot{x} = \alpha x - \beta xy,$$
$$\dot{y} = -\gamma y + \delta xy,$$

where $\alpha, \beta, \gamma, \delta > 0$

(d) Try to find a closed-form solution (not available so far in the literature)

2. Solow Growth model example

(a) Derive the differential equation for the capital-labour ratio (or capital per head), $k = K/L$ using the following

$$\dot{K} = sF(K, L), \quad \frac{\dot{L}}{L} = \lambda.$$

The production function is assumed to have constant returns to scale and be strictly increasing and concave.

(b) Draw its phase diagram.

6.1.6 Budget constraints: where do they come from? (background only)

Assets accumulation is given by the interest income from assets plus wage income minus expenditure. Derive the budget constraint

$$\dot{a}(t) = r(t)\,a(t) + w(t) - e_r(t)$$

formally and also the corresponding interest rate

$$r(t) = \frac{\dot{v}(t) + \pi(t)}{v(t)}.$$

6.1.7 Innovation and growth: optimal demand for varieties

1. Using the setup from Grossman and Helpman given in the lecture, determine the optimal consumption level for households as a function of the price index, defined as $P \equiv \int_0^n p(i)^{1-\varepsilon}\, di$, and expenditure, E.

2. Derive the growth rate of the consumption index.

6.1.8 Innovation and growth: the Keynes-Ramsey rule

Using the indirect utility function, compute the optimal growth rate of expenditure over time (i.e. $\dot{E}(t)/E(t)$), making use of the functional forms (4.2) and (4.4) from the lecture and the result from Exercise 6.1.7 above.

6.1.9 Innovation and growth: optimal behaviour of firms

Derive the optimal price for the intermediate goods firms, using the profit function

$$\pi(i) = p(i)\,x(i) - wl(i)\,,$$

as well as the technology $x(i) = l(i)$, and noting that in equilibrium, the price of good i will depend on the demand for good i, as implied by Exercise 6.1.7.

6.1.10 Equilibrium and reduced form

1. What is the reduced form of the equilibrium of the Grossman and Helpman model? Use the following results from the lecture,

$$\dot{n}(t) = \varphi L_R, \qquad v = w/\varphi,$$

   where $\dot{n}(t)$ is the accumulation of knowledge (or the accumulation of new firms), L_R is the quantity of labour engaged in R&D, v is the value of a firm, and φ is a productivity parameter for R&D workers. The production technology in the economy is equal to labour directed towards building a specific variety for a specific firm i

$$x(i) = l(i).$$

Labour supply L equals labour demand from production and research,

$$L(t) = \int_0^n l(i)di + L_R.$$

2. Draw the phase diagram using the system of equations derived above. Discuss the long-run implications of this model.

6.1.11 Creative destruction: major innovations

1. Final goods firms maximise profits

$$\pi_y(t) = y(t) - p(t)\, x(t).$$

Production takes the form of a Cobb-Douglas function

$$y(t) = \gamma^t x^\alpha(t)\, h^{1-\alpha} = \gamma^t x^\alpha(t)$$

where h is some indivisible production factor normalised to 1, and γ^t is the technological level in period t, where time is an index rather than an argument.

2. What is the demand function of the (many) firms competing in the final good sector for the intermediate good?

3. The intermediate good monopolist maximises profits

$$\pi_x(t) = p(t)\,x(t) - w(t)\,L(t).$$

Production of good x depends linearly on labour, and thus technology for the monopolist is given by

$$x(t) = L(t).$$

What is the optimal price decision for the monopolist?

6.1.12 Optimal saving under Poisson uncertainty (background only)

Consider the objective function

$$U(t) = E_t \int_t^{\infty} e^{-\rho[\tau - t]} u(c(\tau))\, d\tau$$

and the budget constraint

$$da(t) = \{ra(t) + w - p(t)\,c(t)\}\,dt + \beta a(t)\,dq(t),$$

where r and w are constant interest and wage rates, $q(t)$ is a Poisson process with an exogenous arrival rate λ and β is a constant as well. Letting g and σ denote constants, assume that the price $p(t)$ of the consumption good follows

$$dp(t) = p(t)\,[g dt + \sigma dq(t)].$$

1. Derive the Keynes-Ramsey Rule for consumption in the case where p is constant starting from equation 5.24 in the lecture.

2. Now, using the full setup above, derive a rule which optimally describes the evolution of consumption. Deriving this rule in the form of marginal utility, i.e. $du'(c(t))$.

Part II
Unemployment

7 Facts about unemployment

7.1 Definitions

- From population to part-time work

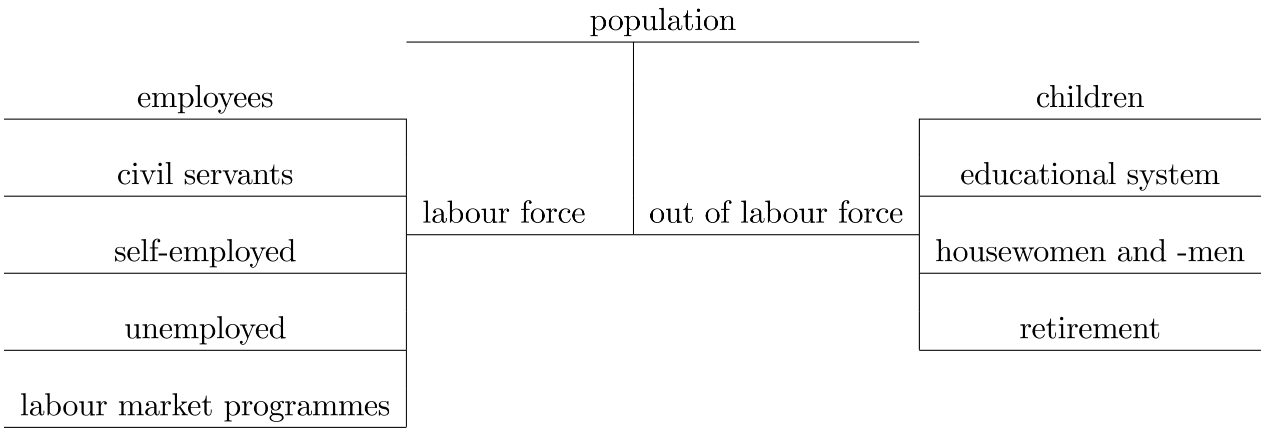


Figure 24 *Classifying a population by economic status*

- Definition of unemployment (OECD-ILO-Eurostat): A worker is unemployed if s/he is

1. without work, that is, were not in paid employment or self employment during the reference period;
2. available for work, that is, were available for paid employment or self-employment during the reference period; and
3. seeking work, that is, had taken specific steps in a specified recent period to seek paid employment or self-employment

7.2 Unemployment stocks

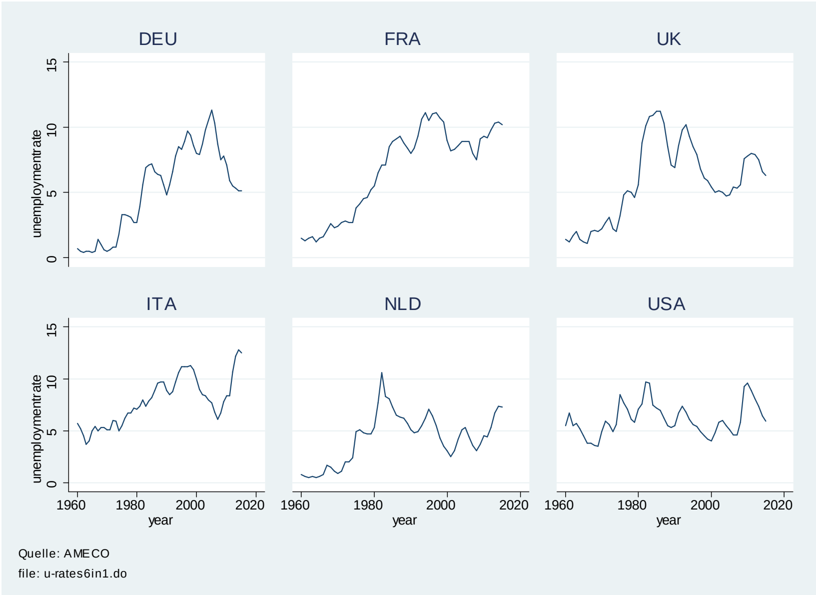


Figure 25 *Unemployment rates from 1960 to today. Source: Slides of Launov and Wälde (2013)*

7.3 Unemployment flows

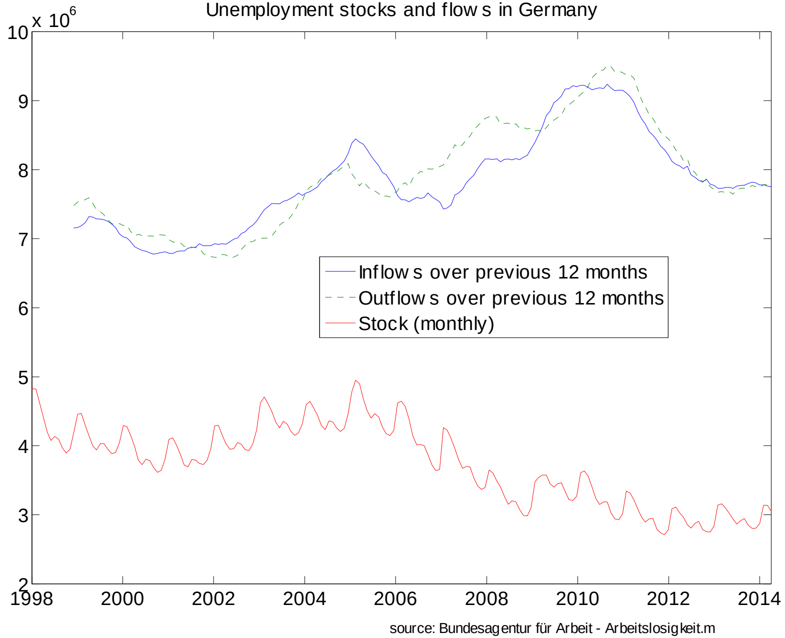


Figure 26 *Stocks and flows on the German labour market*

7.4 Questions for economic theory

- Why do we have unemployment?
  - Is unemployment voluntary?
  - Is it involuntary?
  - (Of primary importance for insurance issues and policy reactions)
- How can we understand both a stock of unemployed and the contemporaneous turnover on the labour market?

8 Matching models of unemployment

8.1 The literature

- Diamond-Mortensen-Pissarides models (Nobel prize in 2010)
- Pissarides (1985) “Short-run Equilibrium Dynamics of Unemployment Vacancies, and Real Wages”
- Pissarides (2000) “Equilibrium Unemployment Theory”, ch. 1
- Rogerson, Shimer and Wright (2005) “Search-Theoretic Models of the Labor Market: A Survey”

8.2 Basic structure

- There is a fundamental job separation process going on in any economy (which is unrelated to the real wage). These separations capture the result of
  - reorganisations of production processes or bankruptcy of firms being caused by
  - new technologies, aggregate business cycle effect, globalisation or other
- Once a worker is unemployed and once a firm has a vacancy
  - there is no such thing as a spot market in the real world
  - finding a job and finding a worker takes time
  - the fundamental reason for this is incomplete information (Stigler, 1961)
- Search processes play an important role on the labour market

8.3 Basic unemployment dynamics

(most closely related to Pissarides, 1985, ch. 1)

8.3.1 An illustration

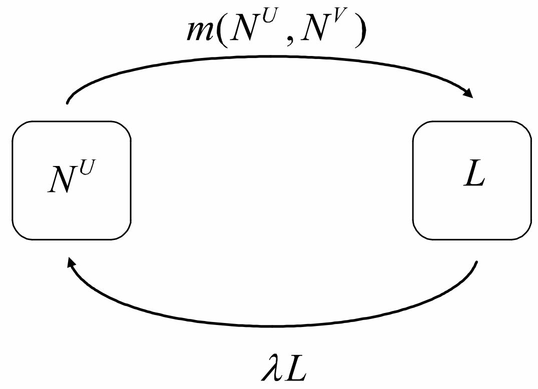


Figure 27 *Labour market flows from employment L to unemployment N^U given a separation rate λ and a matching rate (matching function) $m\left(N^U, N^V\right)$ where N^V is the number of vacancies*

8.3.2 Notation

- Economy consists of a fixed labour force (no labour-leisure choice) N
- A firm either has one vacancy or employs one worker
- Workers are either unemployed or employed

$$N = N^U(t) + L(t)$$

- Unemployment rate

$$u(t) = N^U(t)/N$$

  This implies an employment rate of $1 - u(t)$ - note that all individuals are in the labour force, compare fig. 24

- Vacancy rate

$$v = N^V/N$$

- Job finding rate is the rate with which an unemployed worker finds a job

$$p(\theta) = \frac{m(N^U, N^V)}{N^U} = m(1, \theta) \tag{8.1}$$

  - where the last equality employs the property of constant returns to scale of the matching function $m(.)$ and where
  - the variable

$$\theta = \frac{N^V}{N^U} = \frac{v}{u}$$

    denotes 'labour market tightness' (from the perspective of a firm)

- Job filling rate is the rate with which a vacancy is filled

$$q(\theta) = \frac{m(N^U, N^V)}{N^V} = m\left(\frac{1}{\theta}, 1\right)$$

- Once a firm and a worker have met, they produce output y (a fixed quantity identical for all firm-worker pairs)

8.3.3 The dynamics of the unemployment rate

- The number of unemployed follows (by deriving some mean or by intuition building on figure 27)

$$\frac{d}{dt}N^U(t) \equiv \dot{N}^U(t) = \lambda L(t) - m\left(N^U(t), N^V(t)\right)$$

- What does this equation really mean?

  - All the rates are rates of Poisson processes
  - This implies uncertainty and the model predicts distributions and not deterministic outcomes
  - This equation is, strictly speaking, an equation on the mean of a distribution
  - In most practical senses, this is of no importance for the analyses. We therefore speak of all quantities as if they were deterministic quantities

- Using the definitions from above, we obtain (see Exercise 12.1.1)

$$\dot{u}(t) = \lambda[1 - u(t)] - p(\theta(t))\, u(t) \tag{8.2}$$

- The change of the unemployment rate is determined by

– its current level $u(t)$ and

– labour market tightness $\theta(t)$

- Intermediate illustration - What if $\theta(t)$ was a constant?

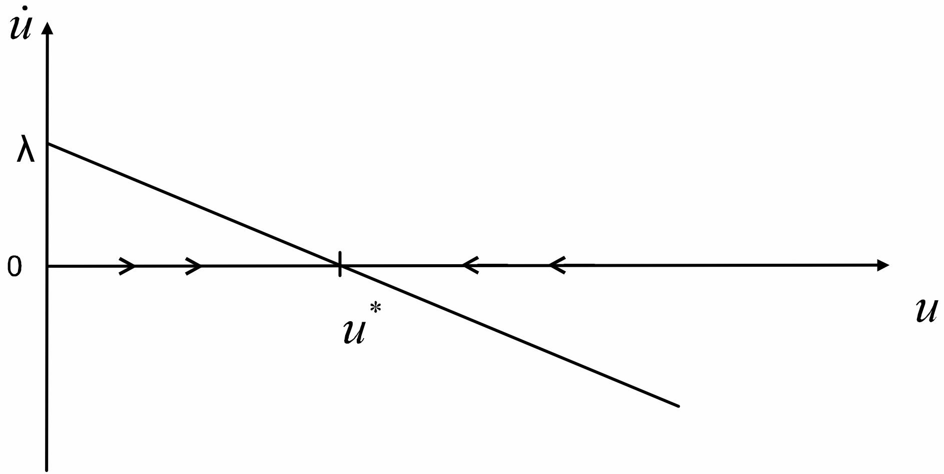


Figure 28 *Phase diagram analysis for* $\dot{u}(t) = \lambda - (\lambda + \mu)\, u(t)$ *where* $\mu \equiv p(\theta)$

- Intermediate illustration - What if $\theta(t)$ was a constant?
  - The unemployment rate has a steady state value at

$$u^* = \frac{\lambda}{\lambda + \mu}$$

  - It rises below and falls above this steady state value
  - It approaches the steady state value at a rate of $\lambda + \mu$

$$u(t) = u^* + (u_0 - u^*)\, e^{-(\lambda+\mu)t}$$

    where u_0 is some initial unemployment level (say, after a shock)
  - This shows how unemployment changes only slowly over time
  - This equation can be used to compute how much time it takes to reduce an economy-wide unemployment rate from, say, 10% to 8%
  - (static models would be unable to make such predictions)

- The next question: How is $\theta(t)$ being determined?
- In order to understand the determination of the number of vacancies (and thereby $\theta(t)$), we need to understand
  - optimal behaviour of workers,
  - optimal behaviour of firms and
  - wage setting
- We now move to Pissarides (1985), i.e. the classic in the field of macro and labour

8.4 The Pissarides (1985) model

8.4.1 Match quality

- Worker and firm meet and observe match quality y
- What does this mean? Output y is random, i.e. drawn (once) from distribution $F(y)$
- Implication: reservation productivity x (compare to reservation wage in pure search) such that a share

$$a(x) \equiv \int_x^\infty f(y)\, dy \tag{8.3}$$

  of contacts actually start producing
- We need to distinguish between a "contact rate" and a "matching rate" (job-finding or job-filling rate)
- Separation rate λ (at individual level) remains unchanged

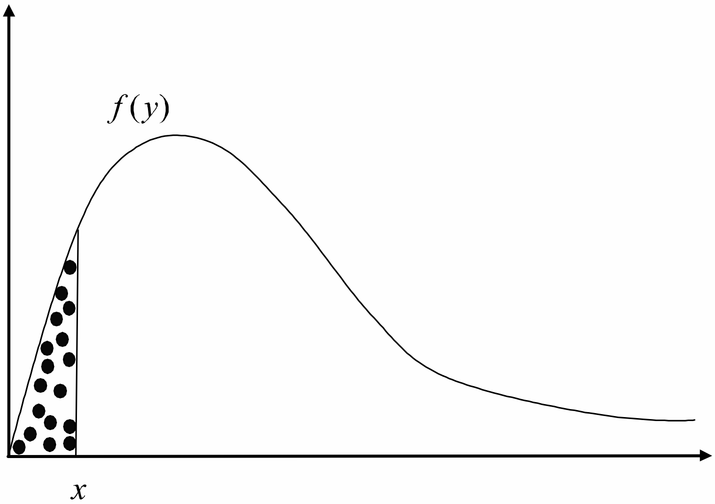


Figure 29 *The density $f(y)$ of productivity y, the reservation productivity x and the share $1-a(x)$ of rejected contacts (dotted area)*

8.4.2 The dynamics of the unemployment rate

- The evolution of the unemployment rate can be derived in the same way as in the setup without stochastic match quality

- Following the same steps, the dynamics of the unemployment rate $u(t)$ is described by

$$\frac{du(t)}{dt} = \dot{u}(t) = \lambda[1 - u(t)] - a(x)\,p(\theta(t))\,u(t) \tag{8.4}$$

  where the unemployment rate

  - rises when (all ceteris paribus) the (individual) separation rate λ is high and the employment rate $1 - u(t)$ is high
  - falls when a large share $a(x)$ of contacts lead to jobs, the job contact rate $p(\theta(t))$ is high and the unemployment rate $u(t)$ is high

- Uncertain match quality introduced a new term, $a(x)$, into the dynamics of $u(t)$

- At some general level, unemployment changes over time only slowly (as before)

- Changes in unemployment are described by this ordinary differential equation. With an initial condition, it can be solved to yield a time path of $u(t)$

8.4.3 Optimal behaviour of workers

- Compared to a model without stochastic job quality, there are two new terms
  - the probability $a(x)$ of finding a good match
  - the expectation in front of the value W_y of being employed
  - (the later is ex-ante unknown as job quality y is unknown)
- The value U of being unemployed is described by the Bellman equation for unemployed workers,
$$\rho U(t) = b + \dot{U}(t) + a(x)\, p(\theta(t)) \left[E W_y(t) - U(t)\right] \tag{8.5}$$
where again new terms are in blue
- The value of being unemployed is determined by
  - how high (utility from) the unemployment benefit b is (individuals are risk neutral and do not save)
  - the change in the value of being unemployed and
  - the expected gain from finding a job where $W_y(t)$ is the value of holding a job of output y

- The value U of being unemployed (again)

$$\rho U(t) = b + \dot{U}(t) + a(x)\, p(\theta(t)) \left[EW_y(t) - U(t) \right]$$

- The expected gain from finding a job is determined by
  - the rate with which an individual finds a job and
  - the difference between the expected value $EW_y(t)$ of being employed and the value $U(t)$ of being unemployed
- Why do we need $EW_y(t)$?
  - The unemployed worker does not know what the match quality y is
  - (S/he only knows that it will be above x, but not *where* above x)
- What is this mean $EW_y(t)$ precisely speaking? It is defined as

$$EW_y(t) \equiv E(W_y(t) | y \geq x) = \frac{\int_x^\infty W_y(t) f(y)\, dy}{1 - F(x)},$$

i.e. the mean conditional on y exceeding the reservation productivity x

- Value $W_y(t)$ of being employed follows

$$\rho W_y(t) = w_y + \dot{W}_y(t) + \lambda [U(t) - W_y(t)] \tag{8.6}$$

- The role of uncertain job quality
  - It does *not* introduce any additional uncertainty here
  - The wage depends on job quality y as does the value $W_y(t)$ of being employed

- The structure of the Bellman equation is unchanged and the usual interpretation can be given: The value of having a job depends on
  - the wage w_y (via the utility it provides given the individual does not save)
  - the change $\dot{W}_y(t)$ in the value of being employed
  - and the expected loss $(U(t) < W_y(t))$ from losing the job

8.4.4 Vacancies and filled jobs

- Firms open vacancies as a function of their expectations about
  - speed of finding a worker
  - joint output
  - implied profit
  - cost k of finding a worker
- Value $V(t)$ of a vacancy

$$\rho V(t) = -k + \dot{V}(t) + a(x) q(\theta(t)) [E J_y(t) - V(t)]$$

  displays the usual determinants like
  - flow costs k
  - change $\dot{V}(t)$ in the value
  - job-filling rate $a(x) q(\theta(t))$ and
  - the value $J_y(t)$ of a job to a firm when joint output is y

- A determinant that now also appears with an expectations operator in this equation

$$\rho V(t) = -k + \dot{V}(t) + a(x)\, q(\theta(t)) \left[E J_y(t) - V(t) \right] \tag{8.7}$$

is the value $J_y(t)$ of employing a worker

  - Expectations need to be formed as this value depends on the uncertain match quality
  - This is the same principle as behind the value $W_y(t)$ of being employed
  - The precise expression displays the truncation

$$E J_y(t) = E(J(y) | y \geq x)$$

    i.e. as expected value of a filled job as average over all matches that persist (for which $y \geq x$)
  - In terms of an integral,

$$E(J(y) | y \geq x) = \frac{\int_x^\infty J_y(t) f(y) dy}{1 - F(x)}$$

- Value $J_y(t)$ of a filled job

$$\rho J_y(t) = y - w(t) + \dot{J}_y(t) + \lambda [V(t) - J_y(t)] \tag{8.8}$$

is again without expectation operator as $V(t)$ has a known value (of zero – see next slide) and y is known once drawn

  – Profits of the firm are given by $y - w(t)$ (output minus wage)
  – Changes of the value of a filled job due to aggregate changes are captured by $\dot{J}_y(t)$
  – The firm's value drops to $V(t)$ at the rate λ

- Free entry into vacancy creation
  - The value of a vacancy equals zero, $V(t) = 0$
  - The Bellman equation (8.7) for vacancy implies

$$EJ_y(t) = \frac{k}{a(x)\, q(\theta(t))}$$

  - The Bellman equation (8.8) for jobs becomes

$$\rho J_y(t) = y - w(t) + \dot{J}_y(t) - \lambda J(t)$$

- Do we not have two equations for $J_y(t)$?
  - No, if the first gives $EJ(t)$ as a function of $\theta(t)$
  - If the second fixes $J_y(t)$, the first fixes $\theta(t)$

8.4.5 Wages

- Both workers and firms have temporary/ local market power once a firm and worker meet
- Wage is determined by some bargaining process – here Nash bargaining
- Technically, we need the contribution of worker and firm to total surplus of match
- Total surplus of match is

$$\begin{aligned} S_y(t) &= W_y(t) - U(t) + J_y(t) - V(t) \\ &= W_y(t) - U(t) + J_y(t) \end{aligned}$$

  i.e. the sum of the gain for the worker and the gain for the firm

- Why is the value of being unemployed not a function of (previous) match quality y?
  - As we assume exogenous benefits
  - In most countries, benefits are given by the benefit replacement rate times the previous wage
  - Extension is treated in exercise 12.1.1

- Nash bargaining with a bargaining power parameter β (of the worker) implies (see exercise 12.1.1)
$$w_y(t) = (1-\beta)\,b + \beta\,[y + \theta(t)\,k]$$
- Nash bargaining takes place initially when firm and worker meet and continuously (at each point in time) - see this as technical simplification compared to bargaining a wage path
- What is the difference to fixed and known match quality?
  - Wage setting equation structurally identical but ...
  - ... wages below reservation wage not visible (not paid in equilibrium)

8.4.6 Job rejection

- Reservation productivity x from the side of the firm

$$J(x) = 0$$

value for the firm from starting production is just as high as from keeping the vacancy

- Reservation productivity x from the side of the worker

$$W(x) = U$$

value from just accepting a job must equal the value of staying unemployed

- Both equations actually fix the same x (see exercise 12.1.2)

$$x = b + \frac{\beta}{1-\beta}\theta k \tag{8.9}$$

  - Reservation productivity x exceeds unemployment benefits when worker bargaining power $\beta > 0$
  - Higher tightness (more vacancies per unemployed worker) increases x

8.4.7 Equilibrium and dynamic adjustment of the unemployment rate

- Reduced form has same structure as without stochastic match quality
  - With stochastic match quality, expectations Eneed to be taken into account
  - With fixed y per match, Ewould drop out

- Equilibrium is described by (see exercise 12.1.2) one equation fixing tightness $\theta(t)$

$$\frac{q'(\theta(t))}{q(\theta(t))}\dot{\theta}(t) = \frac{1-\beta}{k}(Ey-b)\,q(\theta(t)) - \rho - \lambda - \beta p(\theta(t))$$

  and equation (8.4) determining dynamics of unemployment rate $u(t)$

$$\begin{aligned}\dot{u}(t) &= \lambda[1-u(t)] - a(x)\,p(\theta(t))\,u(t) \\ &= \lambda - [\lambda + a(x)\,p(\theta(t))]\,u(t)\end{aligned}$$

- What do these equations tell us?
  - Again, we look at a phase diagram
  - zero-motion line for θ and zero-motion line for u give us steady state
  - arrow-pairs to determine dynamics outside the steady state
  - see next figure (case of fixed match quality y would look identical)

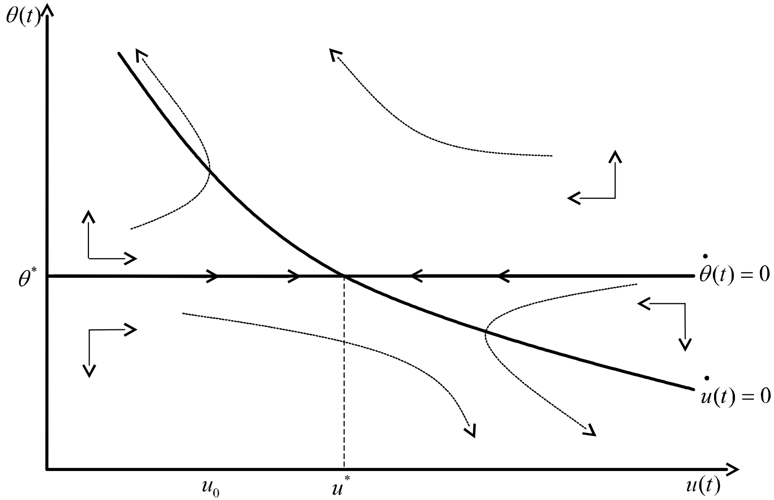


Figure 30 *Phase diagram for the Pissarides textbook matching model*

8.4.8 Response to an output shock

- Now finally, there is a change in productivity, capturing the effect e.g. of a business cycle shock
- What is the effect of a rise in average productivity and costs of vacancy?
- We study a proportional change of productivity by h %

$$y(h) = (1+h)\,y$$

  and would like to understand the reaction of

  - tightness θ
  - the unemployment rate u and
  - the reservation productivity x

- To understand the immediate effects, consider the following figure ...

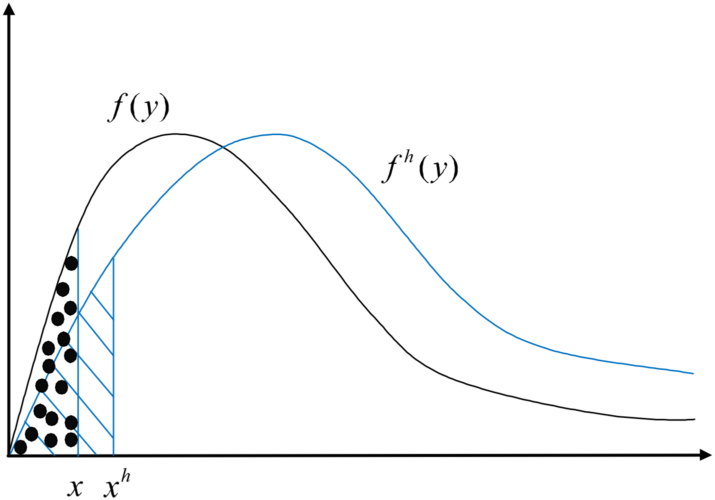


Figure 31 *An increase of productivity by h%, the effect on the reservation productivity x and on the share of acceptable jobs (dashed/ dotted area)*

- Concerning the immediate effects, we would like to understand whether a positive productivity shock ($h > 0$) implies an *instantaneous* increase in unemployment – or whether this happens for a negative productivity shock ($h < 0$) – or never

- When would any change in h have no effects? If a change in h does not affect the share $a(x) \equiv \int_x^\infty f(y)\,dy$ of acceptable jobs from (8.3)

- This is the case if (compare the previous figure) the reservation productivity x changes in exactly the same way as the productivity distribution – in other words, if x also changes by $h\%$

- To understand this

  – look at (8.9) to understand how x depends on (parameters and) tightness θ and

  – understand how h affects θ

- This would show (see exercise 12.1.2) that "the increase in x is not as big as the increase in y and so the acceptance probability rises" (Pissarides, 1985, p. 686)

- In terms of the last figure, a rise in h implies that the dashed area (rejection probability after the improvement of technologies) is smaller than the dotted area (rejection probability at original technology distribution)

- What does this mean for *instantaneous* changes in unemployment?
- A positive technology shock does not lead to any instantaneous changes in unemployment
  - $h > 0$ implies a higher acceptance probability
  - everybody currently in a job remains in the job
- A negative technology shock implies that the acceptance probability goes down
  - A negative h moves the distribution to the left
  - The reservation productivity does not move to the left that quickly (reduces by less than h percent)
  - Some workers that were in "just acceptable" jobs, i.e. jobs just above x will prefer unemployment to staying in this job after a reduction by h%
- A negative technology shock therefore leads to an *instantaneous* increase in unemployment
- ... and this is exactly what we see on the next figure

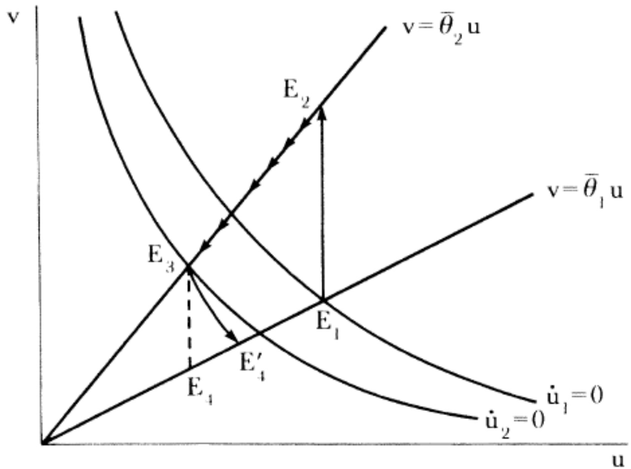


Figure 32 *The effect of multiplicative technology shocks on the dynamics of vacancy rate v (not θ on axis!) and unemployment u (Pissarides, 1985, fig. 1)*

8.5 What have we learned?

Let us return to initial questions

- Why are individuals unemployed?
  - separation rate, job offer rejection
  - finding a job takes time (search process) due to incomplete information
- Is unemployment voluntary or involuntary?
  - involuntarily: exogenous job separation
  - voluntary unemployment: reject offers, quit job after negative technology shock
- What are reasons for involuntary unemployment?
  - exogenous separation process
  - no detailed modelling (in this paper) of biased technological change, globalisation and other

- This model is the “workhorse” i.e. the standard model on which 100s of other papers build on
  - Analysis of economic growth
  - Looking at endogenous separation processes (idiosyncratic technology shocks)
  - Analysis of tax policy
  - Analysis of labour market policy
  - Framework for structural estimation
  - much much more ...

9 Search unemployment

9.1 The basic search model

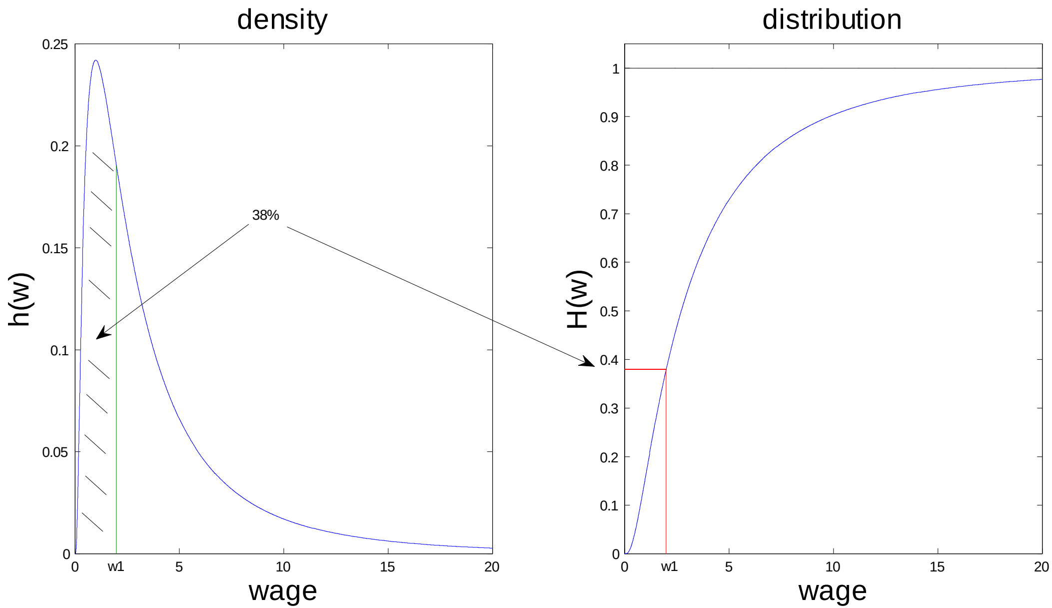
→ see Cahuc Zylberberg (2004, ch. 3)

9.1.1 The basic idea

- reason for search: lack of information about job availability and the wage paid per job
- setup
  - look at one unemployed worker
  - receives unemployment benefits
  - intensity of search is *not* chosen
  - can *not* look for another job once employed
  - stationary environment
- question we can ask: which wage is accepted once an offer is made?

9.1.2 Expected utility once employed

- unemployed does not know which wage will be offered once a job is found
- knows that all wages are drawn from the same (continuous cumulative) distribution $H(w)$ with density $h(w)$



density
h(w)
0.25
0.2
0.15
0.1
0.05
0
0
w1
5
10
15
20
wage
38%
distribution
H(w)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
w1
5
10
15
20
wage

9.2

9.1.2 Expected utility once employed (cont'd)

- Worker is risk neutral
  - risk-neutrality means that the utility function is linear in income
  - here: utility function is given by real labour market income (wage or benefit)
- When employed the worker loses the job
  - at (separation) *rate* $s > 0$, meaning that
  - the *probability* to lose the job over period of time of length dt is given by sdt
  - (again Poisson process in continuous time)
- Real instantaneous interest rate r (= time preference rate)
- This gives us value of being employed (Bellman equation for employed worker - compare (8.6))

$$rV_e = w + s\left[V_u - V_e\right]$$

- Rewrite this for later purposes as

$$V_e(w) - V_u = \frac{w - rV_u}{r + s} \tag{9.1}$$

9.1.3 The optimal search strategy

- We assume job searcher only meets one employer at a time
- An offer consists of a fixed wage w
- Choice between 'accept' or 'reject'
- Optimality criterion: is V_e or V_u higher?
- Accept $\Leftrightarrow V_e(w) > V_u$, which from (9.1) is the case if and only if

$$w > rV_u \equiv x \tag{9.2}$$

- We have thereby defined the reservation wage x
- Intuition why ever reject
  - Disadvantage from accepting a job consists in the inability to further look for jobs (as there is *no* on-the-job search)
  - Employee is stuck with wage w for a potentially long time
  - It might be better to reject and hope for better offer (with higher wage w)

9.1.4 The discounted expected utility (value function) of a job seeker

- Arrival rate of job: λ
- λ reflects labour market conditions, personal characteristics (age, educational background), effort (time and carefulness put into writing applications, not modeled here)
- Unemployment benefits b and opportunity costs of search c give instantaneous utility when unemployed, $z \equiv b - c$
- Value of being unemployed (Bellman equation for unemployed worker)

$$rV_u = z + \lambda \int_x^\infty \left[V_e(w) - V_u\right] h(w)\, dw \tag{9.3}$$

- (same logic as for Bellman equation (8.5) in matching model)

9.1.5 Reservation wage

- Bellman equation for unemployed worker can also be used to obtain an expression for reservation wage
- Remember the definition of the reservation wage in (9.2), $x = rV_u$
- Employing (9.3) and some further steps (see exercise 12.1.3) yields

$$x = z + \lambda \frac{\int_x^\infty (w - x) h(w) dw}{r + s} \tag{9.4}$$

- Interpretation as above for rV_u, apart from $r + s$ in denominator
  - $\frac{\pi}{r}$ is the present value (when discounting with r) of receiving income (profits) π forever
  - $\frac{\pi}{r+s}$ is the present value of receiving π as long as it randomly stops at rate s
  - hence $\frac{\int_x^\infty (w-x)h(w)dw}{r+s}$ is the present value of receiving a wage above x until exit rate s hits
  - z is received instantaneously as a flow and λ is the arrival of a job offer

9.1.6 Hazard rates and average duration in unemployment

- What is hazard rate (exit rate with which an individual leaves unemployment)?

$$\text{exit rate} = \lambda \left[1 - H(x)\right]$$

  where λ is the job offer rate and $1 - H(x)$ is the probability of accepting a job

- What is the average duration T_u in unemployment?

$$T_u = \frac{1}{\lambda \left[1 - H(x)\right]} \tag{9.5}$$

  (using a standard property of Poisson processes, duration is exponentially distributed)

- This allows for many policy analyses concerning determinants of unemployment rate (duration in unemployment to be more precise).

9.2 Non-stationary search

9.2.1 Institutional background

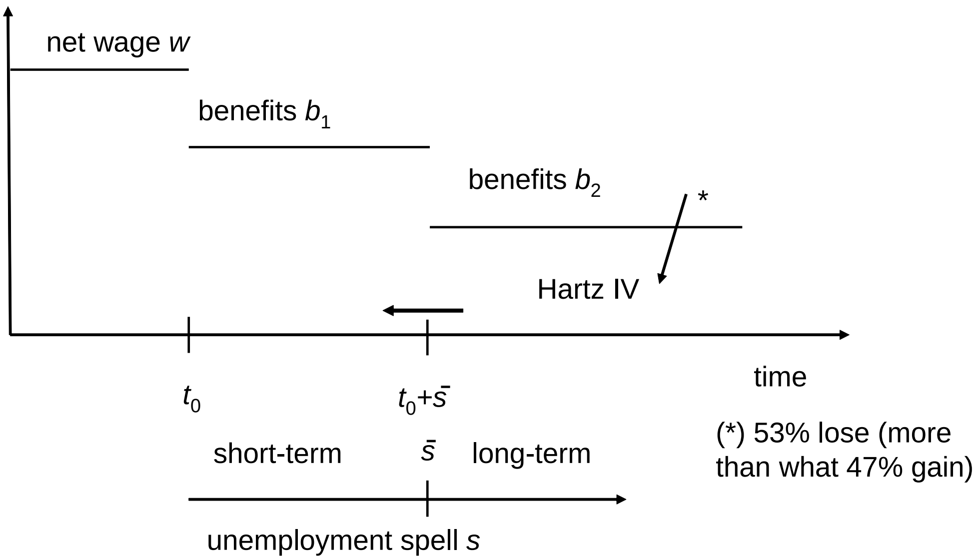


Figure 33 *Unemployment benefit payments in Germany (and many OECD countries (and the Hartz reform))*

9.2.2 Empirical background

- In matching and in search model
  - exit rates from unemployment at the individual level were independent of duration in unemployment
  - They were even constant in steady states
- It is well-known, however, that exit rates are duration dependent
  - Take a flow sample of entry into (un)employment (each month of 1997 and 1998), giving us total of 743 individuals (Launov and Wälde, 2013, 2016)
  - exit rates of one cohort strongly falls over time

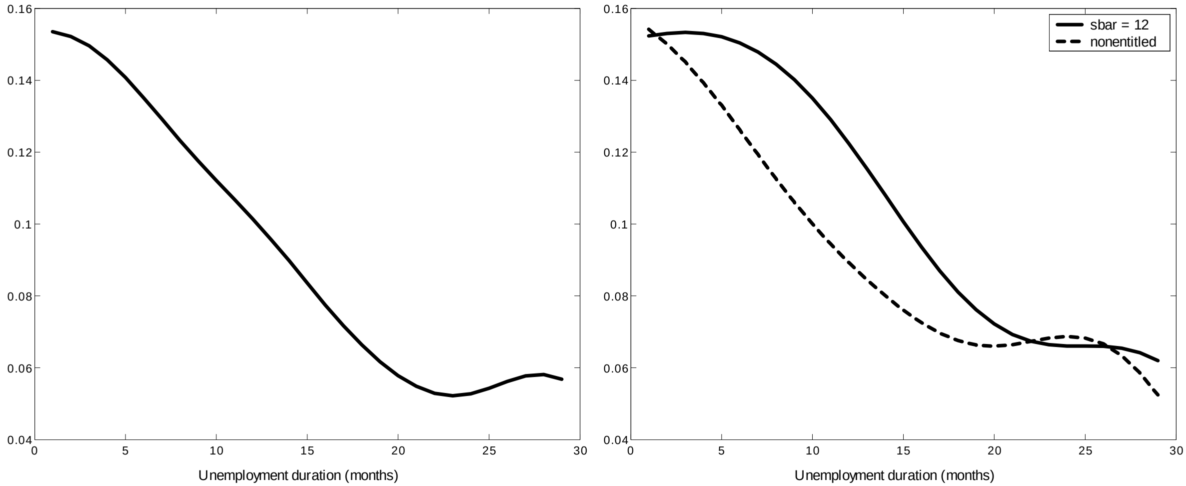


Figure 34 *Non-parametric exit rates from unemployment for one cohort*

- Do these exit rates result from *individually* falling exit rates or are they due to a *composition* effect?

- In the latter case, individuals with high exit rates leave first, leaving a pool with a lower *average* exit rate

9.2.3 Two-tier unemployment benefit systems

- Question and (some) literature

  - How do individuals behave when unemployment benefits are time-variant?
  - Early analysis is by Mortensen (1977) for this simplest possible case of non-stationarity
  - Luckily, this is also the institutionally most relevant case (compare literature on optimal unemployment insurance)
  - Estimation of a pure search model: van den Berg (1990)
  - Equilibrium analysis and structural estimation in a matching model: Launov and Wälde (2013, 2016)
  - See Cahuc and Zylberberg (2004) for a broader overview

- Institutional background

$$b(s) = \begin{cases} b_{UI} & 0 \leq s \leq \bar{s} \\ b_{UA} & \bar{s} < s \end{cases} . \tag{9.6}$$

  Unemployment benefits depend on duration s in unemployment (not on calender time)

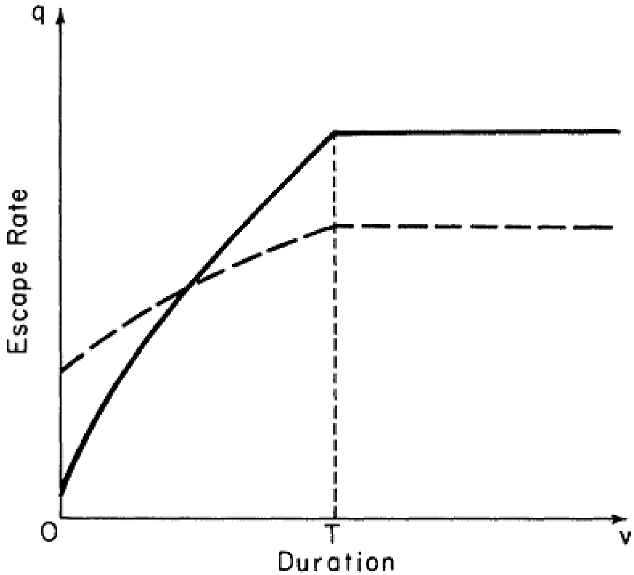


Figure 35 *Optimal 'escape rates' (exit rates) from unemployment when unemployment rates fall at T (from Mortensen, 1977)*

- Let us understand where this figure comes from
- Value of being employed
  - Once the worker has a job, he never loses it
  - State of being employed is an “absorbing state”
  - The wage is constant
  - Hence, value of being employed is constant and given by (see exercise 12.1.4)

$$\rho V(w) = u(w) \tag{9.7}$$

    where

    * ρ is time preference/ interest rate and
    * $u(w)$ is utility from consumption paid by wage income w

- Value of being unemployed

$$\rho V\left(b\left(s\right),s\right)=\max_{\phi(s)}\left\{u\left(b\left(s\right),\phi\left(s\right)\right)+\frac{dV\left(b\left(s\right),s\right)}{ds}+\mu\left(\phi\left(s\right)\right)\left[V\left(w\right)-V\left(b\left(s\right),s\right)\right]\right\} \tag{9.8}$$

  - Compared to (8.5) and (9.3) this Bellman equation is
    - simpler as wage w is non-random
    - more general as benefits $b\left(s\right)$ change over time and effort $\phi\left(s\right)$ can be chosen
  - Value $V\left(b\left(s\right),s\right)$ of being unemployed changes over duration s in unemployment (not calender time) as
    - benefits change and
    - point $\bar{s}$ in time where benefits change comes closer

- First-order condition

  - The loss from more search effort equals the gain from higher arrival rate,

$$\frac{\partial}{\partial\phi\left(s\right)}u\left(b\left(s\right),\phi\left(s\right)\right)+\mu'\left(\phi\left(s\right)\right)\left[V\left(w\right)-V\left(b\left(s\right),s\right)\right]=0 \tag{9.9}$$

- Optimal behaviour
  - In order to make further progress (numerically or with estimation), we need functional forms
  - Instantaneous utility function of an unemployed worker

$$u\left(b\left(s\right),\phi\left(s\right)\right)=\frac{b\left(s\right)^{1-\sigma}-1}{1-\sigma}-\phi\left(s\right) \tag{9.10}$$

  - The exit rate from unemployment is given by (fundamental of the model)

$$\mu\left(\phi\left(s\right)\right)=\eta\phi\left(s\right)^{\alpha}, \tag{9.11}$$

    where η is a productivity parameter and α determines the elasticity of the arrival rate with respect to effort $\phi\left(s\right)$
  - Optimal effort implied by (9.9) can then be expressed as

$$\phi\left(s\right)=\left\{\alpha\eta\left[V\left(w\right)-V\left(b\left(s\right),s\right)\right]\right\}^{1/(1-\alpha)} \tag{9.12}$$

- Maximized Bellman equation for unemployed worker ...
  - ... starting from (9.8) and employing (9.12) ...
  - ... can be written as a differential equation and reads

$$\frac{dV(b(s),s)}{ds} = \rho V(b(s),s) - \frac{b(s)^{1-\sigma}-1}{1-\sigma} - \frac{1-\alpha}{\alpha}(\alpha\eta\theta^{\alpha})^{1/(1-\alpha)}[V(w)-V(b(s),s)]^{1/(1-\alpha)} \tag{9.13}$$

  - How do we proceed from there?
    * How do we solve differential equations again (or difference equation, same principle)?
    * What do we need in order to get a unique solution (see Wälde, 2012, ch. 4.1.2)?
  - We need a boundary condition
    * Where do we get this?

- The value at or after $\bar{s}$ where individual receives b_{UA}
  - Bellman equation reads

$$\rho V(b_{UA}) = \max_{\phi(s)} \{u(b_{UA}, \phi(s)) + \mu(\phi(s))[V(w) - V(b_{UA})]\}$$

  - The worker lives in a stationary world, there is no reason to assume that the value of being unemployed changes, hence $\frac{dV(b_{UA},s)}{ds} = 0$ and $V = V(b_{UA})$
  - First-order condition for optimal effort ϕ in analogy to (9.12)

$$\phi = \{\alpha\eta[V(w) - V(b_{UA})]\}^{1/(1-\alpha)}$$

  - Given the constant value $V(w)$ of being employed from (9.7), effort is constant as well for $s \geq \bar{s}$
  - These equations jointly fix $V(b_{UA})$ and ϕ, given $V(w)$ (see matlab code from Exercise 12.1.4)
- What is the value of being unemployed at $\bar{s}$ where benefits $b(s)$ are discontinuous?
  - The value is continuous

$$V(b_{UI}, \bar{s}) = V(b_{UA}, \bar{s}) = V(b_{UA}). \tag{9.14}$$

  - (see Launov and Wälde, 2013, for some discussion)

- Let us now return to $V\left(b\left(s\right),s\right)$
  - What is the value of being unemployed in the short run, i.e. for $s<\bar{s}$?
  - We solve the ODE in (9.13) for $V\left(b_{UI},s\right)$, employing $V\left(b_{UA}\right)$ as initial condition
  - We summarize the entire system on the next slide

* Compute (for a given set of exogenous parameters)

$$u(w) = \frac{w^{1-\sigma} - 1}{1-\sigma}$$
$$V(w) = \frac{u(w)}{\rho}$$

* Solve for $V(b_{UA})$ (fixpoint problem, e.g. fzero in matlab, employing additional exogenous parameters)

$$\phi = \{\alpha\eta [V(w) - V(b_{UA})]\}^{1/(1-\alpha)}$$
$$\mu(\phi) = \eta\phi^{\alpha}$$
$$u(b_{UA}, \phi) = \frac{b_{UA}^{1-\sigma} - 1}{1-\sigma} - \phi$$
$$\rho V(b_{UA}) = u(b_{UA}, \phi) + \mu(\phi)[V(w) - V(b_{UA})]$$

* Take $V(b_{UI}, \bar{s})$ as boundary condition to solve ODE for $V(b_{UI}, s)$

$$V(b_{UI}, \bar{s}) = V(b_{UA})$$
$$\frac{dV(b_{UI}, s)}{ds} = \rho V(b_{UI}, s) - \frac{b_{UI}^{1-\sigma} - 1}{1-\sigma} - \frac{1-\alpha}{\alpha} (\alpha\eta\theta^{\alpha})^{1/(1-\alpha)} [V(w) - V(b_{UI}, s)]^{1/(1-\alpha)}$$

- Parameter values for numerical solution

time preference rate	ρ	2.4% (annual)
		$\approx$ 0.198% (monthly)
risk aversion	σ	0.8
search effort elasticity	α	0.4
search productivity	η	0.02
wage	w	1200 EUR (monthly)
unemployment insurance payments	b_{UI}	65% of w
unemployment assistance payments	b_{UA}	45% of w
entitlement to UI payments	$\bar{s}$	18 (months)

- (taken from the structural estimation by Launov and Wälde, 2013)

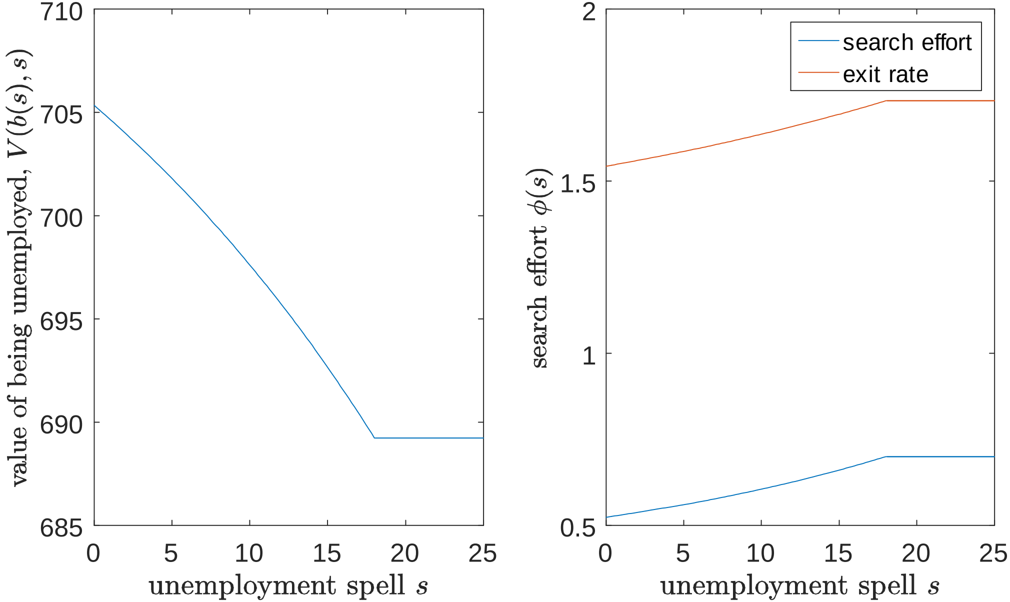


Figure 36 *Value $V(b(s), s)$ of being unemployed from (9.13) in left panel and search effort (9.12) and exit rate from unemployment (9.11) in right panel*

9.3 What have we learned?

- Can we explain downward sloping exit rates of cohorts as shown in fig. 34?
  - Non-stationary unemployment benefits were an idea
  - Fig. 35 and fig. 36 show that it was a good idea but it does not work
  - Effect of non-stationary (falling) unemployment benefits actually imply that exit rates should go up (as there is highest search incentive with lowest exit rate)
- What is then the reason for exit rates that depend negatively on duration in unemployment?
- Maybe Bayesian learning can help

10 Search with Bayesian learning

10.1 Bayesian learning

10.1.1 The standard expected utility framework

- Standard homo oeconomicus works with *objective* probabilities

  - With a discrete underlying random variable , he/ she/ it maximizes expected utility (EU) given objective probabilities

    $$EU = \Sigma_{i=1}^{n} p_i u_i$$

    where there are n states i of the world, u_i is utility in state i and p_i are *objective* probabilities

  - When there is a continuum of states i, homo oeconomicus would maximize

    $$EU = \int_{i^{\min}}^{i^{\max}} u(i) f(i) \, di$$

    where the state of the world ranges from $i^{\min}$ to $i^{\max}$, $u(i)$ is utility in state i and $f(i)$ is the density of i

- Imagine probabilities p_i are *not* objectively know (pretty realistic assumption)

- What should individual maximize? Probabilities are not available ...

10.1.2 Subjective expected utility theory

- When objective probabilities are not known, individual ...
  - ... just employs subjective ones, according to the approach of subjective expected utility (SEU) theory
  - ... uses heuristics, is boundedly rational or other, according to other approaches
- With SEU, objective function now reads

$$EU = \Sigma_{i=1}^{n} p_i^s u_i$$

or

$$EU = \int_{i^{\min}}^{i^{\max}} u(i) f^s(i) \, di$$

where p_i^s are now subjective probabilities and $f^s(i)$ is the subjective density function

- Immediate issue with SEU-theory
  - Can individuals not learn?
  - Why do they not collect information (consciously or not) and adjust their subjective beliefs?
- This is where Bayesian learning starts

10.1.3 Bayes' theorem

- Some background: Conditional probabilities
  - Consider two discrete random variables X and Y
  - The probability distribution of X is given by π_X with probabilities $\pi_X(x)$, i.e. $\pi_X(x)$ is the probability that the random variable X takes the realization of x,

$$\text{Prob}(X = x) \equiv \pi_X(x) \text{ for } x \in \{x_1, x_2, ..., x_n\}$$

  - Likewise

$$\text{Prob}(Y = y) \equiv \pi_Y(y) \text{ for } y \in \{y_1, y_2, ..., y_n\}$$

  - The joint probability distribution of X and Y is denoted by π_{XY} with probabilities

$$\text{Prob}(X = x \wedge Y = y) \equiv \pi_{XY}(x, y) \text{ for } y \in \{x_1, ..., x_n\} \times \{y_1, ..., y_n\}$$

- Some background: Conditional probabilities (cont'd)
  - The probability distribution for X conditional on Y is then denoted by $\pi_{X|Y}$ and $\pi_{X|Y}(x|y)$ is the probability that $X = x$ conditional on $Y = y$. It is defined as

$$\pi_{X|Y}(x|y) \equiv \frac{\pi_{XY}(x,y)}{\pi_Y(y)} \tag{10.1}$$

  - In words, $\pi_{X|Y}(x|y)$ is the probability of x given that y has realized
  - A standard example takes X to be body length and Y gender of a person: $\pi_{X|Y}(x|\text{male})$ is the probability of body length x when considering only male individuals

$$\pi_{X|Y}(x|\text{male}) \equiv \frac{\pi_{XY}(x,\text{male})}{\pi_Y(\text{male})}$$

- See e.g. DeGroot (1970, 2004, ch. 2.4) and beyond, for more background (e.g. on continuous random variables)

- Bayes theorem for two discrete random variables
  - Bayes' theorem says
$$\pi_{X|Y}(x|y) = \frac{\pi_{Y|X}(y|x)\,\pi_X(x)}{\pi_Y(y)} \tag{10.2}$$
  - Proof is straightforward from employing the definition in (10.1) of the conditional probability twice (see e.g. DeGroot, 1970, 2004, ch. 2.4)
- Interpretation
  - A frequent interpretation is in terms of 'learning' or updating 'beliefs'
  - Imagine X is a measure of agreeableness
    - This is one of the 'big 5' personality measures from personality psychology (see e.g. John et al., 2008, Almlund et al., 2011)
    - Agreeableness measures how much individuals are trusting, pro-social, compassionate, warm, friendly, honest and considerate
    - It can take various values on a discrete/ continuous scale
  - To be precise and simple, imagine one can either be agreeable ($x = 1$) or not ($x = 0$)
  - Further, $Y \in \{1, 2, ..., n\}$ is the number of (good) friends a person has

- Interpretation (cont'd)
  - The individual *does not know* whether he is agreeable but he holds a *subjective prior distribution* about X
  - This prior distribution is given by

$$\begin{aligned} \text{Prob}_{\text{subj}}(X = 1) &= p_{\text{subj}} \\ \text{Prob}_{\text{subj}}(X = 0) &= 1 - p_{\text{subj}} \end{aligned}$$

    such that p_{subj} is the subjective belief to be 'a nice guy'
  - The *objective* probabilities to have y friends conditional on x are given by $\pi_{Y|X}(y|1)$ and $\pi_{Y|X}(y|0)$
  - By the 'law of total probability' aka 'total probability theorem' (e.g. Wackerly, Mendenhall and Scheaffer, 2008, ch. 2.10), the unconditional probability to have y friends is given by

$$\pi_Y(y) = p_{\text{subj}}\pi_{Y|X}(y|1) + (1 - p_{\text{subj}})\,\pi_{Y|X}(y|0) \tag{10.3}$$

    where the probability to be agreeable is the subjective one (as objective one is not available)
  - Now imagine a person has y friends. What is the probability to be agreeable, given the observation on y?

- Interpretation (cont'd)
  - Bayes' theorem (10.2) provides the answer as

$$\pi_{X|Y}(1|y) = \frac{\pi_{Y|X}(y|1)}{\pi_Y(y)} p_{\text{subj}}$$

    where $\pi_{X|Y}(1|y)$ is the probability to be agreeable ($x = 1$) given the observation to have y friends

- Updating beliefs
  - This probability $\pi_{X|Y}(1|y)$ can be called the subjective belief to be agreeable ($x = 1$) given the observation to have y friends,

$$p_{\text{subj}}(y) \equiv \pi_{X|Y}(1|y)$$

  - Writing this as

$$p_{\text{subj}}(y) = \frac{\pi_{Y|X}(y|1)}{\pi_Y(y)} p_{\text{subj}} \qquad (10.4)$$

    shows us what *updating of beliefs* means
    * The individual starts with a subjective probability p_{subj} to be a nice guy

* He has knowledge on objective probability $\pi_{Y|X}(y|1)$ and on $\pi_Y(y)$ from (10.3)
* S/he makes the observation to have y friends
* This allows to adjust or update the belief from p_{subj} to $p_{\text{subj}}(y)$

- An alternative interpretation
  - imagine X stands for 'sufficiently intelligent for getting a Bachelor degree'
  - and let Y stand for the exams already passed in a Bachelor programme
  - Then the subjective belief to be sufficiently intelligent after y successful exams is given by the same equation as in (10.4),

$$p_{\text{subj}}(y) = \frac{\pi_{Y|X}(y|1)}{\pi_Y(y)} p_{\text{subj}}$$

- See also Zwillinger and Kokoska (2000, ch. 3.3.9) or DeGroot (1970, ch. 8.9) for further examples

- Bayes' theorem for three random variables

(for general interest only – not for lecture)

Now let us look at the case of three random variables X, Y and Z. Distributions are then defined as above and the joint distribution is denoted by π_{XYZ} with probabilities $\pi_{XYZ}(x, y, z)$. Conditional probabilities are then e.g.

$$\pi_{X|Y,Z}(x|y,z) = \frac{\pi_{XYZ}(x,y,z)}{\pi_{YZ}(y,z)}, \tag{10.5}$$

$$\pi_{Y|X,Z}(y|x,z) = \frac{\pi_{XYZ}(x,y,z)}{\pi_{XZ}(x,z)}. \tag{10.6}$$

Expressed in words for the first equation, the probability that $X = x$ conditional on $Y = y$ and $Z = z$ is given by the probability that $X = x$ and $Y = y$ and $Z = z$, as displayed in the numerator, divided by the joint probability that $Y = y$ and $Z = z$ as displayed in the denominator.

This allows us to express a generalized version of Bayes theorem as

$$\pi_{X|Y,Z}(x|y,z) = \frac{\pi_{XZ}(x,z)}{\pi_{YZ}(y,z)} \pi_{Y|X,Z}(y|x,z). \tag{10.7}$$

The proof follows the same principle as in the two-dimensional case. Divide (10.5) by (10.6) and find (10.7).

10.1.4 Economic literature on Bayesian learning

- Seminal papers written by DeGroot
  - Bayesian Analysis and Duopoly Theory (Cyert and DeGroot, 1970)
  - Rational Expectations and Bayesian Analysis (Cyert and DeGroot, 1974)
  - See also his textbook (DeGroot, 1970, 2004)
- Various economic applications
  - An individual's valuation of an unknown product or decision (Rothschild, 1974)
    - Valuation may vary over time
    - It will vary in the light of new evidence
  - The decision between buying a commodity with a certain quality and a commodity with an uncertain quality (Kihlstrom, 1974a, b)
    - The consumer as a Bayesian learner can invest into information about the uncertain commodity prior to his consumption decision
    - Consumers will neither buy information about very cheap or very expensive products, nor about products which are expected to be of very low or very high quality

- Choice of a firm between different production technologies (Tonks, 1983)
  * One technology yields known, the other yields unknown returns
  * Two period setup with no gains from learning in the second period (as there is not third where information would be useful) but gains from learning in the first period
  * Buy more of uncertain technology (as compared to known technology) in first period due to gains from learning (active learner as opposed to passive learner)
- Strategic aspects in games (Keller et al., 2005, Keller and Rady, 2010)
  * Bayesian learning in continuous time with Poisson-uncertainty
  * Two-armed bandit problem where one arm yields a safe and the other a risky payoff
  * Game of strategic experimentation, where each player faces an identical two-armed bandit problem and observes the other players' actions and outcomes

• Learning in search and matching models
  - Studied by Launov Wälde (2013, 2015)
  - Unemployed workers do not know their search productivity
  - Optimal search behaviour becomes duration-dependent (see below)

10.1.5 Learning with continuous random variables

- So far we described uncertainty by a discrete random variable
- Now assume we would like to work with a continuous random variable
- Assume that there is a true distribution for the random variable that is normal
  - The spread is known, say, we know the variance σ_ε^2
  - The mean of the true distribution is unknown
  - As the mean is unknown, we need to make distributional assumptions about this unknown mean – this is the prior
  - Assume this prior distribution is normal as well with initial mean η_0 and variance σ_0^2
  - Assume further that we make n observations u_i drawn from the true distribution of this random variable

- The starting point of Bayesian learning for a normal true distribution and a normal prior (left and middle figure)
- With information, the right figure becomes relevant

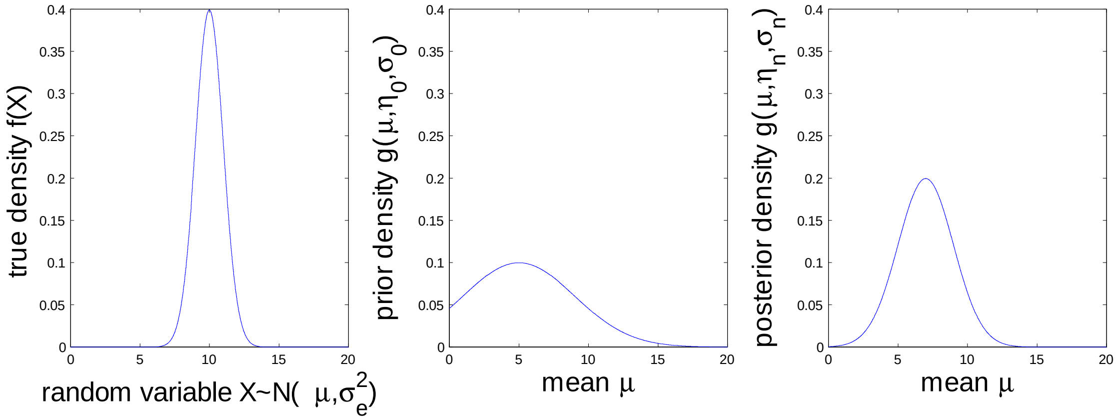


Figure 37 *The true distribution of X with mean μ and variance σ_ε^2, the prior normal density of the mean (with mean η_0 and variance σ_0^2) and the posterior after n observations (with mean η_n and variance σ_n)*

- Given a normal true distribution and a normal distribution of the prior, we then we have the following (see deGroot, 1970, Theorem 1, ch. 9.5, note that his precision is the inverse of the variance, see ch. 4.7)

Proposition The posterior distribution of the unknown mean is
(i) a normal distribution
(ii) characterized by a mean $\eta_n$that follows

$$\eta_n = \frac{n\sigma_0^2}{\sigma_\varepsilon^2 + n\sigma_0^2}\overline{u}_n + \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + n\sigma_0^2}\eta_0, \tag{10.8}$$

where $\overline{u}_n = \frac{1}{n}\Sigma_{i=1}^n u_i$
(iii) and variance σ_n^2 given by

$$\sigma_n^2 = \frac{\sigma_\varepsilon^2\sigma_0^2}{\sigma_\varepsilon^2 + n\sigma_0^2}. \tag{10.9}$$

- How to understand this theorem
  - There is an individual that “has no clue” about e.g. his/her “social agreeableness”, about his/her IQ or other
  - He knows that IQs are normally distributed, he knows the variance of the distribution but not the mean
  - He forms a prior distribution about the mean, assumes this mean to be normally distributed and posits (exogenously) a mean η_0 and a variance σ_0^2
  - Now there are n occasions which provide information about the individuals IQ, i.e. there are n drawings from the true distribution
  - This allows to compute the posterior distribution, i.e. the mean and variance of the distribution for the unknown mean are adjusted

- How to understand this theorem (cont'd)
  - The mean converges to the true mean,

$$\lim_{n\to\infty} \eta_n = \overline{u}_n = \frac{1}{n}\Sigma_{i=1}^{n} u_i,$$

    and the variance of the prior converges to zero,

$$\lim_{n\to\infty} \sigma_n^2 = 0.$$

  - In other words, with an infinity of observations, an individual becomes certain about the mean of his/ her distribution

- Applications
  - This theorem can be (and has been) used for many applications
  - Learning processes about all kind of economic and non-economic parameters or variables can easily be included in models
  - One can distinguish between passive learning (observations come exogenously) and active learning (individual chooses the amount of informations – which might be costly)

10.1.6 Bayesian learning in continuous time

- Discrete time models
  - Structures for Bayesian learning we got to know so far are very useful for static models or models in discrete time
  - One can imagine, for example, the number of observations n as being drawn in a period t, $n = n_t$
  - One can then renormalize t to zero and start from the updating in (10.8) and (10.9) again
  - This is a structure employed in many papers in the literature
- But what about updating in continuous time models?
  - Consider an example taken from Launov and Wälde (2013, 2016)
  - The derivation is similar but not identical to Keller, Rady and Cripps (2005)
  - The contents is similar but in its modelling not identical to Keller and Rady (2010)

- Consider a search and matching setup, in particular a worker who is unemployed
  - Consider an unemployed worker who has been unemployed for a duration of s
  - Let the worker hold a subjective probability that he is a good searcher $(X = 1)$
  - This is a random variable $X \in \{0, 1\}$ with the subjective probability being denoted by $p(s)$
$$\text{Prob}(X = 1) \equiv p(t)$$
  - The event 'transition into employment' is the second random variable $Y(s) \in \{0, 1\}$ where 0 means 'no transition' and 1 means 'transition into employment' in s
  - If the individual is a good searcher, the rate with which he finds a job is $\eta_1 > \eta_0$ where η_0 is the rate with which the worker finds a job when he is a bad searcher
  - We can now ask what the level of $p(s)$ is, or how it changes, given observations on transitions into employment (or not)

- Starting with Bayes' theorem (10.2),

$$\pi_{X|Y}(x|y) = \frac{\pi_{Y|X}(y|x)}{\pi_Y(y)} \pi_X(x),$$

we replace the various probabilities as follows:

  - The current belief $p(s)$ corresponds to the unconditional probability $\pi_X(x)$,i.e. $\pi_X(1) \equiv p(s)$
  - The posterior is the level of the belief (for $X = 1$) after the realization, i.e. after having observed to still be without job ($Y = 0$), $\pi_{X|Y}(1|0) = p(s + ds) = p(s) + dp(s)$
  - The unconditional probability of Y is the probability of the event 'no transition' ($Y = 0$), $\pi_Y(0) = p(s)[1 - \mu_1 ds] + (1 - p(s))[1 - \mu_0 ds]$
  - The conditional probability of Y is the probability of 'no transition' conditional on being a good searcher, $\pi_{Y|X}(y|1) = 1 - \mu_1 ds$

- Putting everything together gives

$$p(s) + dp(s) = \frac{1 - \mu_1 ds}{p(s)[1 - \mu_1 ds] + (1 - p(s))[1 - \mu_0 ds]} p(s) \tag{10.10}$$

- The interpretation, again
  - the posterior probability $p(s) + dp(s)$ for having a high search productivity on the left-hand side, is given by
  - the prior $p(s)$ times
  - the likelihood of the evidence
    * i.e. the event 'no jump' conditional on having high search productivity, $1 - \mu_1 ds$
    * divided by the unconditional probability that there is no jump

- Simplifying (10.10) yields evolution of the belief (see exercise 12.1.5)

$$\dot{p}(s) = -p(s)(1 - p(s))(\mu_1 - \mu_0) \tag{10.11}$$

- What does this equation tell us?
  - As $\mu_1 > \mu_0$, the unemployed worker believes less and less that he is good at searching, $\dot{p}(s) < 0$
  - This is no surprise as the evidence collected at each instant is 'no transition took place'
  - The only meaningful conclusion is that one's search productivity is low

10.2 Non-stationary search with Bayesian learning in equilibrium

- We now combine Bayesian learning with non-stationary unemployment benefits
- Why again?
  - Empirical puzzle of fig. 34 is still unexplained
  - Fig. 35 and fig. 36 show that non-stationary (falling) unemployment benefits lead to an increase in exit rates
- What is the idea here?
  - Individuals do not know how good they are at finding a job
  - When they do not find a job, they correct their belief downwards that they are good at finding a job
  - Lower (subjective) job finding probability decreases search effort
  - Exit rate falls in duration of unemployment

10.2.1 The model

- Mortensen-Pissarides type matching model extended for
  - time-dependent unemployment benefits
  - endogenous effort
  - risk-averse households and
  - an endogenous negative duration dependence resulting from subjective beliefs
  - Individuals differ by their search productivity type $\chi \in \{0, 1\}$
    * Search productivity type χ is not known
    * Individuals can learn their type over time in a Bayesian fashion
  - (simplified version of Launov and Wälde, 2013, 2016)

10.2.2 Workers

- As before, unemployment payments $b(s)$ are spell-dependent

$$b(s) = \begin{cases} b_{UI} & 0 \leq s \leq \bar{s} \\ b_{UA} & \bar{s} < s \end{cases}$$

- An unemployed worker finds a job according to a time-inhomogeneous Poisson process with an *objective* arrival rate $\mu(\phi(s)\theta, \chi)$
  - This rate is also called job-finding rate, hazard rate or exit rate out of unemployment
  - This rate depends on
    * effort $\phi(s)$ an individual exerts to find a job
    * labour market conditions captured by labour market tightness $\theta \equiv V/(N-L)$
    * an individual's search productivity type χ

- We let individuals behave like (passive) Bayesian learners that update some belief $p(s)$ that χ equals one
  - The information for the update stems from the duration of unemployment
  - Individuals will base their decisions on a *subjective* arrival rate $\mu(\phi(s)\theta, p(s))$
- This setup allows us to obtain endogenous falling exit rates at the individual level

- Outcome of the time-varying exit rate [background – how to undertake structural estimation]

  - Endogenous distribution of unemployment duration with density (e.g. Ross, 1996, ch. 2)

    $$f(s,\chi) = \mu(\phi(s)\theta,\chi)\, e^{-\int_0^s \mu(\phi(u)\theta,\chi)du}, \tag{10.12}$$

    one for each value of χ

  - Densities are crucial for various purposes including the estimation of model parameters

  - It is endogenous to the model, as the exit rate $\mu(\phi(s)\theta,\chi)$ is determined by the optimizing behaviour of workers and firms

  - This is the basis of structural estimation

10.2.3 Optimal behaviour

- Bellman equation for being employed
  - Households are infinitely lived and do not save
    * Time preference rate ρ
    * Present value of having a job $V(w)$ depends on the current endogenous wage w only
    * Employed workers enjoy instantaneous utility $u(w)$
    * A worker-firm match can be interrupted at rate λ (time-homogenous Poisson process)
    * Unemployed workers always receive insurance payments b_{UI} for the length of $\bar{s}$ (full re-entitlement)
    * Value of being unemployed at duration $s = 0$ is $V(b_{UI}, 0)$
  - This leads to a Bellman equation for the employed worker of

$$\rho V(w) = u(w) + \lambda [V(b_{UI}, 0) - V(w)] \tag{10.13}$$

- Bellman equation for being unemployed
  - Bellman equation for the unemployed worker reads

$$\rho V(b(s), s) = \max_{\phi(s)} \left\{ u(b(s), \phi(s)) + \frac{dV(b(s), s)}{ds} + \mu(\phi(s)\theta, p(s)) [V(w) - V(b(s), s)] \right\} \tag{10.14}$$

  - As in (9.8) explicit state variables are benefits $b(s)$ and duration s
- The instantaneous utility flow of being unemployed, $\rho V(b(s), s)$, is given by three components
  - instantaneous utility resulting from consumption of $b(s)$ and effort $\phi(s)$
  - a deterministic change of $V(b(s), s)$ as the value of being unemployed changes over time
  - a stochastic change that occurs at the *subjective* job-finding rate $\mu(\phi(s)\theta, p(s))$

- An optimal choice of effort $\phi(s)$ for (10.14) requires

$$u_{\phi(s)}(b(s), \phi(s)) + \mu_{\phi(s)}(\phi(s)\theta, p(s))[V(w) - V(b(s), s)] = 0 \tag{10.15}$$

  where subscripts denote partial derivatives. It states that the utility loss resulting from increasing search effort must be equal to expected utility gain due to higher effort

- Given functional forms for
  - utility as in (9.10), $u(b(s), \phi(s)) = \frac{b(s)^{1-\sigma}-1}{1-\sigma} - \phi(s)$
  - the subjective exit rate as an extension to (9.11),

$$\mu(\phi(s)) = \eta(s)\phi(s)^{\alpha}$$

    where now $\eta(s)$ stands for a subjective search productivity

$$\eta(s) \equiv (1 - p(s))\eta_0 + p(s)\eta_1 \tag{10.16}$$

- the first-order condition for effort (10.15) reads

$$\phi(s) = \{\alpha\eta(s)\theta^{\alpha}[V(w) - V(b(s), s)]\}^{1/(1-\alpha)} \tag{10.17}$$

  holding for both short- and long-term unemployed

10.2.4 Findings

- How to get them
  - Solve numerically for a steady state similar to the procedure on slide 9.2.3 ....
  - ... extended for a change of belief that as a generalized version of (10.11) reads

$$\frac{dp(s)}{ds} = -p(s)(1-p(s))(\mu(\phi(s)\theta,1) - \mu(\phi(s)\theta,0)) \tag{10.18}$$

- The findings
  - (see next figure)
  - exit rates now fall individually ...
  - Alternative interpretation of empirically falling exit rates from fig. 34 beyond composition effect

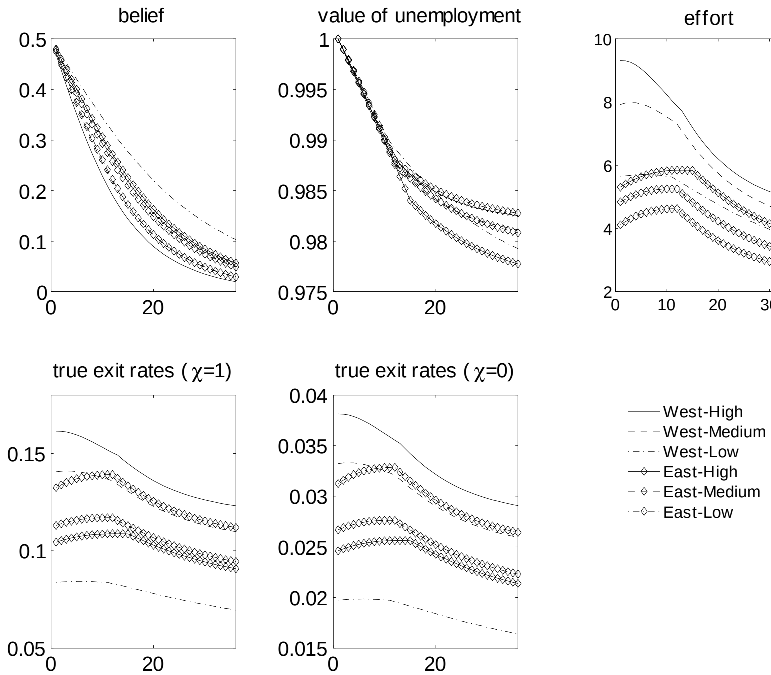


Figure 38 *Effort and exit rates under Bayesian learning*

10.3 What have we learned?

- We studied Bayesian learning

  - Bayesian learning studies how subjective beliefs are updated over time
  - Updating is the mathematically/ statistically correct way to update subjective beliefs
  - (human behaviour generally differs)
  - It is a useful first step in a departure from full-information structures where individual knows probabilities objectively

- We applied Bayesian learning to non-stationary search environments

  - We saw that downward sloping exit rates can be explained
    * The mechanism is a true duration dependence of exit rates due to 'discouraged workers'
    * Their belief that they have a high search productivity falls over time
    * This complements the composition effect
  - [model allows to capture exit rates of unemployed workers also quantitatively in a satisfatory way]

11 Search and matching and self-insurance

11.1 Why should we care?

- The question – version 1
  - There is no explicit financial market in SaM models
  - Households are risk-neutral – this does not allow to study questions of insurance in a meaningful way
  - Should not households want to self-insure against income fluctuations? We should allow for precautionary saving
  - Various authors have studied theses issues relatively recently
    - Lentz (2009) Job search with savings (and estimation)
    - Krusell, Mukoyma and Sahin (2010) study saving in Pissarides matching model
    - Lise (2013) studies saving in search model with wage distribution (and estimation)

- The question – version 2
  - We observe wealth distributions which are considered to be too unequal
  - Indirect evidence from the success of Piketty's (2014) "Capital in the Twenty-First Century"
  - Direct evidence from Norton and Ariely (2011) "Building a Better America" (see https://youtu.be/QPKKQnijnsM for a very emotional but informative video)
  - Desired wealth distribution (Gini coefficient: 0.2) is more equal than believed wealth distribution (0.51) and the actual one (0.76)
  - This is of interest to very many economists very recently
    * (Bewley-Hugget-Aiyagari idiosyncratic risk model in labour income)
    * Benhabib, Bisin and Zhu (2007) allow for interest rate risk
    * Gabaix et al. (2015) study distributional dynamics
    * For much more background on wealth distributions (and their dynamics), see introduction to Wälde (2016) or GSEFM seminar announcement on wealth inequality

- The plan
  - Look at a model which allows to study the effects of unemployment benefits on unemployment when individuals save
  - Ask whether such a model can quantitatively replicate observed wealth distributions
  - Closest to this setup is Lise (2013) who has a pure search foundation, i.e. a wage distribution including a reservation wage
  - We abstract from wage distribution here to make central insights clearer

• How do wealth distributions look like?

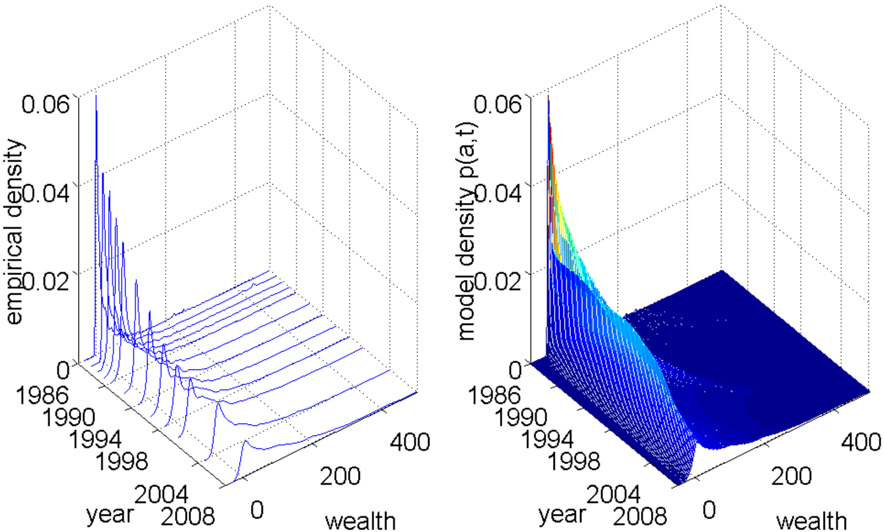


Figure 39 *The density of wealth for the NLSY 79 cohort (left figure)*

11.2 The equations behind Lorenz and Gini [background]

- What was the Gini coefficient again?

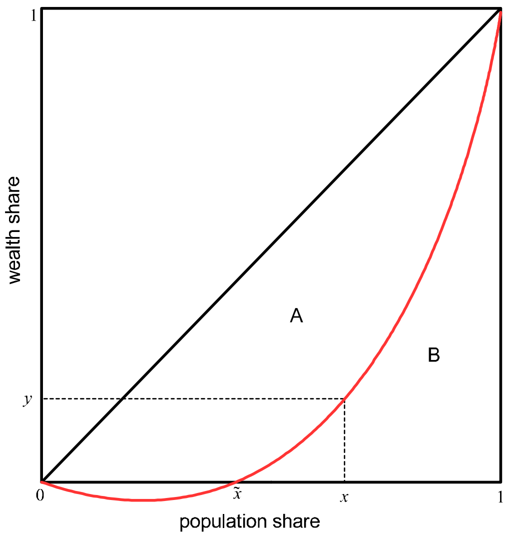


Figure 40 *Ilustrating the Gini coefficient for a (red) Lorenz curve with negative values*

- The equations behind the figure
  - (all taken from appendix of Khieu and Wälde, 2018)
  - The Gini coefficient is given by $G = \frac{A}{A+B}$
  - The areas B and A are given by, respectively,

$$B = \int_{\tilde{x}}^{1} y dx, \qquad A = \frac{1}{2} - B - \int_{0}^{\tilde{x}} y dx = \frac{1}{2} - \int_{\tilde{x}}^{1} y dx - \int_{0}^{\tilde{x}} y dx,$$

where $\tilde{x}$ satisfies $y(\tilde{x}) = 0$ and where $\int_0^{\tilde{x}} y dx < 0$ which requires the minus sign in the expression for A. Thus,

$$G = \frac{\frac{1}{2} - \int_0^1 y dx}{\frac{1}{2} - \int_0^{\tilde{x}} y dx}.$$

- What was the Lorenz curve again?
  - Let there be a continuous random variable a standing for levels of wealth. The pdf is $f(a)$ over the support $[a_{\min}, a_{\max}]$. The Lorenz curve tells us that $x\%$ of the population hold $y\%$ of total wealth
  - The share of population holding $\tilde{a}$ or less is given by

$$x(\tilde{a}) = \int_{a_{\min}}^{\tilde{a}} f(a)da,$$

    where $\tilde{a} \in [a_{\min}, a_{\max}]$.
  - The wealth share owned by x is given by

$$y(\tilde{a}) = \frac{\int_{a_{\min}}^{\tilde{a}} af(a)da}{\int_{a_{\min}}^{a_{\max}} af(a)da}$$

  - The Lorenz curve is then constructed by mapping x into y (see figure above). For each $x \in [0, 1]$ there is one and only one value $y \in [0, 1]$. Thus, y is a function of x.

11.3 The structure

11.3.1 Labour income

- Labour income $z(t)$ stochastically jumps between two income levels, w and b

$$dz(t) = [w - z(t)]\, dq_\mu(t) + [b - z(t)]\, dq_s(t) \tag{11.1}$$

- The arrival rate of the separation process q_s is $s > 0$

- The arrival rate of the Poisson process q_μ related to job finding is $\mu > 0$

- Note that q_μ and q_s can always jump (independently of $z(t)$)

- But a jump has an effect only when $z(t)$ is "in the other state"

  - When $z(t) = w$, the equation reads

$$dz(t) = [w - w]\, dq_\mu(t) + [b - w]\, dq_s(t)$$

  - When q_μ jumps, it has no effect
  - Similar reasoning for $z(t) = b$ and q_s
  - $z(t)$ therefore jumps between w and b (assuming an initial condition $z(0) \in \{w, b\}$)

11.3.2 The individual

- Standard intertemporal and instantaneous (CRRA) utility functions

$$U(t) = E_t \int_t^\infty e^{-\rho[\tau - t]} u(c(\tau))\, d\tau \tag{11.2}$$

$$u(c(\tau)) = \frac{c(\tau)^{1-\sigma} - 1}{1-\sigma}$$

- Budget constraint for wealth $a(t)$

$$da(t) = \{ra(t) + z(t) - c(t)\}\, dt \tag{11.3}$$

- Constraint for labour income $z(t)$ from (11.1)

$$dz(t) = [w - z(t)]\, dq_\mu(t) + [b - z(t)]\, dq_s(t)$$

- Compared to optimal saving under interest uncertainty in (5.21) with (5.22) on slide 5.33 where
  - uncertainty resulted from an uncertain interest rate
  - uncertainty now comes from uncertain labour income

- The maximization problem
  - Maximize the objective function (11.2) subject to the wealth constraint (11.3) and the income constraint (11.1)
  - (Adjusted) General Bellman equation as starting point from (5.23) reads

$$\rho V(a(t), z(t)) = \max_{c(t)} \left\{ u(c(t)) + \frac{1}{dt} E_t dV(a(t), z(t)) \right\} \tag{11.4}$$

  - Computing the differential $dV(a(t), z(t))$ using the CVF (5.20), taking the constraints (11.3) and (11.1) into account and forming expectations, yields (with time argument suppressed)

$$\rho V(a, z) = \max_{c} \left\{ \begin{array}{c} u(c) + [ra + z - c] V_a(a, z) \\ +s[V(a, b) - V(a, w)] + \mu [V(a, w) - V(a, b)] \end{array} \right\} \tag{11.5}$$

  - The right hand side of this BE has the usual suspects
    * instantaneous utility
    * smooth change in the value function $V(a, z)$ due to saving
    * jumps in the value function due to changes in labour income

- How does this Bellman equation differ ...
  - ... from matching Bellman equations we already saw e.g. in sect. 8.4 or
  - ... from search Bellman equations in sect. 9.1?
  - ... from, as one example, the Bellman equation (8.6),

$$\rho W_y(t) = w_y + \dot{W}_y(t) + \lambda\left[U(t) - W_y(t)\right],$$

for being employed?

- Set $z = w$ and thereby $\mu(w) = 0$ in (11.5) to get

$$\rho V(a,w) = \max_c \left\{u(c) + [ra + w - c]\, V_a(a,w) + s\left[V(a,b) - V(a,w)\right]\right\}$$

- Differences consist in
  - the absence of aggregate (deterministic changes) – $\dot{W}_y(t)$ is missing
  - the presence of savings and the corresponding shadow price $V_a(a,w)$ of wealth
  - the presence of a meaningful maximization problem

- Optimal behaviour given our Bellman equation in (11.5)

$$u'\left(c\left(a,z\right)\right)=V_a\left(a,z\right). \tag{11.6}$$

- The first-order condition equates marginal utility from consumption with the shadow price of wealth
  - By the budget constraint (11.3), one unit of consumption costs one unit of wealth
  - Hence, the marginal utility from consumption is identical to the present value increase in overall utility due to an additional unit of wealth

- Where do we stand now?
  - As discussed earlier after (3.14), we could stop with analytical steps and use numerical solution methods
  - We could solve two equations (11.5) and (11.6) for control variables/ policy functions $c\left(a,w\right)$ and $c\left(a,b\right)$ and value functions $V\left(a,w\right)$ and $V\left(a,b\right)$ numerically
  - Again, further analytical steps yield more economic insights, however

11.3.3 Optimal behaviour

- Keynes-Ramsey rules (see ex. 12.1.6)
  - There is one version when worker is in state 'employed' and ...
  - ... one when in state 'unemployed'

- Employed worker

$$\frac{dc\left(a_w,w\right)}{c\left(a_w,w\right)}=\left\{\frac{r-\rho}{\sigma}+\frac{s}{\sigma}\left\{\left[\frac{c\left(a_w,w\right)}{c\left(a_w,b\right)}\right]^{\sigma}-1\right\}\right\}dt-\left[1-\frac{c\left(a_w,b\right)}{c\left(a_w,w\right)}\right]dq_s$$

  - $s=0$: deterministic world with $\dot{c}/c=\left(r-\rho\right)/\sigma$
  - $s>0$: consumption growth is faster for employed worker due to ...

- Precautionary saving

$$\left[\frac{c\left(a_w,w\right)}{c\left(a_w,b\right)}\right]^{\sigma}=\frac{u'\left(c\left(a_w,b\right)\right)}{u'\left(c\left(a_w,w\right)\right)}$$

  - high growth of consumption if marginal utility in unemployment state is high relative to employment state
  - consumption smoothing by accumulating wealth fast

- Unemployed worker

$$\frac{dc(a_b,b)}{c(a_b,b)} = \left\{\frac{r-\rho}{\sigma} - \frac{\mu}{\sigma}\left\{1-\left[\frac{c(a_b,b)}{c(a_b,w)}\right]^{\sigma}\right\}\right\} dt + \left[\frac{c(a_b,w)}{c(a_b,b)} - 1\right] dq_{\mu}$$

  - $\mu = 0$: deterministic world with $\dot{c}/c = (r-\rho)/\sigma$ (as before)
  - $\mu > 0$: consumption growth is *slower* for unemployed worker due to ...

- Anticipation of future higher income ("post-cautionary dissaving")

  - choose a higher consumption level than in certain world with b forever
  - wealth growth and thereby consumption growth is lower
  - same fundamental motive as in employment state

- What to do with these equations?
  - They look nice and make economically sense
  - But how to proceed from there?
  - Can we get more out of them?
  - Trick: exploit "piece-wise deterministic nature" of Poisson processes
  - Question: how does consumption evolve when worker *remains in one state*?
- For times between jumps, above equations become

$$\frac{\dot{c}(a_w, w)}{c(a_w, w)} = \frac{r-\rho}{\sigma} + \frac{s}{\sigma}\left\{\left[\frac{c(a_w, w)}{c(a_w, b)}\right]^{\sigma} - 1\right\} \tag{11.7}$$

$$\frac{\dot{c}(a_b, b)}{c(a_b, b)} = \frac{r-\rho}{\sigma} - \frac{\mu}{\sigma}\left\{1 - \left[\frac{c(a_b, b)}{c(a_b, w)}\right]^{\sigma}\right\} \tag{11.8}$$

- Unfortunately, this is not a 2-dimensional ODE system as $c(a_w, w)$ in first equation is not $c(a_b, w)$ in second equation as $a_w(t) \neq a_b(t)$
- Way out: "time-elimination method"

- What is the time-elimination method? [background]
  - (Name is due to or used by Sala-i-Martin, see Mulligan and Sala-i-Martin, 1991, NBER Technical Working Paper 116)
  - Consider the Solow-growth model with optimal saving
  - The system from (3.3) and (3.4) reads

$$\frac{dK(t)}{dt} = \dot{K}(t) = Y(K(t), L) - \delta K(t) - C(t)$$
$$\frac{dC(t)}{dt} = \dot{C}(t) = \left(\frac{\partial Y(K(t), L)}{\partial K(t)} - \delta - \rho \right) C(t) / \sigma$$

  - Now "divide" equations and get

$$\frac{\frac{dC(t)}{dt}}{\frac{dK(t)}{dt}} = \frac{dC}{dK} = \frac{\left(\frac{\partial Y(K,L)}{\partial K} - \delta - \rho \right) C / \sigma}{Y(K, L) - \delta K - C}$$

- What is the time-elimination method? [background cont'd]

$$\frac{\frac{dC(t)}{dt}}{\frac{dK(t)}{dt}} = \frac{dC}{dK} = \frac{\left(\frac{\partial Y(K,L)}{\partial K} - \delta - \rho\right) C/\sigma}{Y(K,L) - \delta K - C}$$

  - We now have one ODE for C in K (and no longer in time)
  - This ODE describes the slopes of the trajectories in the phase diagram
  - With an appropriate initial condition $C(K_0)$, there is a unique solution (the saddle path)

- Advantage of this method?

  - Not clear for simple systems that can easily be solved in time
  - Differential equations become fewer in number (one less) but more complex
  - But sometimes further insights are possible

- Time elimination method for matching and saving
  - Write the above system (11.7) and (11.8) as

$$\dot{c}(a_w, w) = \left(\frac{r-\rho}{\sigma} + \frac{s}{\sigma} \left\{ \left[\frac{c(a_w, w)}{c(a_w, b)} \right]^{\sigma} - 1 \right\} \right) c(a_w, w)$$
$$\dot{c}(a_b, b) = \left(\frac{r-\rho}{\sigma} - \frac{\mu}{\sigma} \left\{ 1 - \left[\frac{c(a_b, b)}{c(a_b, w)} \right]^{\sigma} \right\} \right) c(a_b, b)$$

and "add" corresponding budget constraints from (11.3), one for each $z \in \{w, b\}$

$$\dot{a}_w(t) = ra_w(t) + w - c(a_w(t), w) \tag{11.9}$$
$$\dot{a}_b(t) = ra_b(t) + b - c(a_b(t), b) \tag{11.10}$$

  - Now eliminate time ("divide the equations") and get

$$\frac{dc(a_w, w)}{da_w} = \frac{\frac{r-\rho}{\sigma} + \frac{s}{\sigma} \left\{ \left[\frac{c(a_w,w)}{c(a_w,b)} \right]^{\sigma} - 1 \right\}}{ra_w + w - c(a_w, w)} c(a_w, w)$$
$$\frac{dc(a_b, b)}{da_b} = \frac{\frac{r-\rho}{\sigma} - \frac{\mu}{\sigma} \left\{ 1 - \left[\frac{c(a_b,b)}{c(a_b,w)} \right]^{\sigma} \right\}}{ra_b + b - c(a_b, b)} c(a_b, b)$$

– Now eliminate time and get (copied from previous slide)

$$\frac{dc\left(a_w, w\right)}{da_w} = \frac{\frac{r-\rho}{\sigma} + \frac{s}{\sigma}\left\{\left[\frac{c(a_w,w)}{c(a_w,b)}\right]^\sigma - 1\right\}}{ra_w + w - c\left(a_w, w\right)} c\left(a_w, w\right)$$

$$\frac{dc\left(a_b, b\right)}{da_b} = \frac{\frac{r-\rho}{\sigma} - \frac{\mu}{\sigma}\left\{1 - \left[\frac{c(a_b,b)}{c(a_b,w)}\right]^\sigma\right\}}{ra_b + b - c\left(a_b, b\right)} c\left(a_b, b\right)$$

– We have obtained a system where time is gone

* the exogenous variable (the variable with respect to which the differential equation is solve) is now wealth a
* and no longer time t

– As a consequence, we can replace a_w and a_b by a – and we have an ODE system!

$$\frac{dc\left(a, w\right)}{da} = \frac{\frac{r-\rho}{\sigma} + \frac{s}{\sigma}\left\{\left[\frac{c(a,w)}{c(a,b)}\right]^\sigma - 1\right\}}{ra + w - c\left(a, w\right)} c\left(a, w\right) \tag{11.11}$$

$$\frac{dc\left(a, b\right)}{da} = \frac{\frac{r-\rho}{\sigma} - \frac{\mu}{\sigma}\left\{1 - \left[\frac{c(a,b)}{c(a,w)}\right]^\sigma\right\}}{ra + b - c\left(a, b\right)} c\left(a, b\right) \tag{11.12}$$

11.4 Consumption and wealth dynamics

11.4.1 Reduced form and phase diagram

- Reduced form
  - We have now obtained a two-dimensional ODE system
  - But what are its properties?
  - Let's try to get some intuition by ...
  - ... a phase diagram
- What do we need again for a phase diagram?
  - Need to know what to plot on axes (simple: a and c)
  - Need zero-motion lines
  - Need arrows of motion

- Zero-motion line for consumption
  - Under which conditions does consumption of a worker rise (over time!)?
  - Consumption an employed worker rises if and only if $c(a_w, w)$ relative to $c(a_w, b)$ is sufficiently high,

$$\frac{dc(a_w, w)}{dt} \geq 0 \Leftrightarrow \left[\frac{c(a_w, w)}{c(a_w, b)}\right]^{\sigma} \geq 1 - \frac{r-\rho}{s} \tag{11.13}$$

  - For the unemployed worker, consumption rises if and only if $c(a_b, b)$ relative to $c(a_b, w)$ is sufficiently high,

$$\frac{dc(a_b, b)/dt}{c(a_b, b)} \geq 0 \Leftrightarrow \left[\frac{c(a_b, b)}{c(a_b, w)}\right]^{\sigma} \geq 1 - \frac{r-\rho}{\mu}$$

  - straightforward consequence of Keynes-Ramsey rules

- What do these conditions tell us?
  - They depend on parameters and interest rate (which is exogenous in partial equilibrium)
  - It is useful to distinguish between low, intermediate and high regime

$$r \in \{(0, \rho], (\rho, \rho + \mu), [\rho + \mu, \infty)\}$$

  - In this lecture (and Lise, 2013 and precautionary saving literature), $r < \rho$
  - For more, see "explosive regime" in Benhabib and Bisin (2016) or Wälde (2016)

- How can we understand these conditions for $r < \rho$?
  - They define threshold levels a_w^* and a_b^*
  - Considering the example of a worker being employed, we can write (11.13) as

$$\frac{dc(a_w, w)}{dt} \geq 0 \Leftrightarrow \left[\frac{c(a_w, w)}{c(a_w, b)}\right]^\sigma \geq 1 - \frac{r-\rho}{s} \Leftrightarrow a_w < a_w^*$$

where a_w^* is defined such that

$$\left[\frac{c(a_w^*, w)}{c(a_w^*, b)}\right]^\sigma = 1 - \frac{r-\rho}{s} \tag{11.14}$$

  - In words: consumption of an employed worker grows, when wealth is below threshold level a_w^*, otherwise it falls
  - Also in words (some derivation skipped): Consumption of unemployed worker fro $r < \rho$ always falls

- Zero motion lines for wealth
  - This follows standard procedure
  - We start from (11.3) or from (11.9) and (11.10)
  - For both states the worker can be in, wealth rises if consumption is sufficiently small

$$\dot{a}_w(t) \geq 0 \Leftrightarrow c(a_w(t), w) \leq r a_w(t) + w \tag{11.15}$$

$$\dot{a}_b(t) \geq 0 \Leftrightarrow c(a_b(t), b) \leq r a_b(t) + b \tag{11.16}$$

- Phase diagram
  - All of this now allows us to draw a phase diagram (for $r < \rho$)

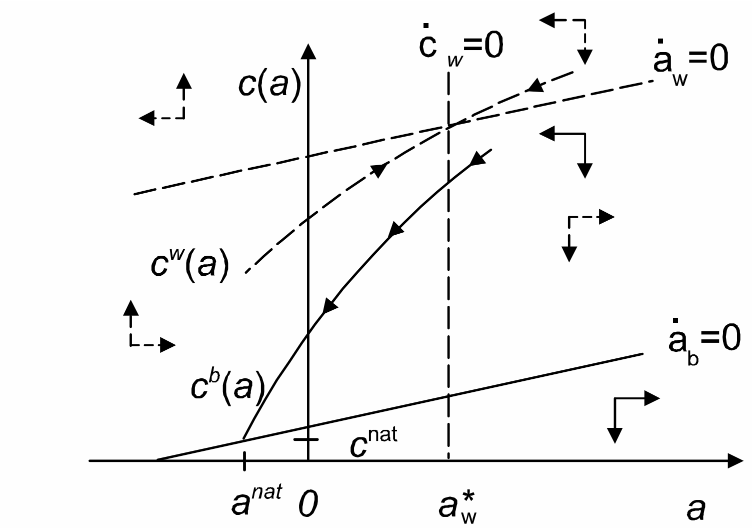


Figure 41 *Wealth-consumption dynamics for a worker transitioning between two states (w and b)*

- What the phase diagram shows us
  - A zero-motion line for consumption when employed as a vertical line at a_w^*
    * which is still only implicitly known from (11.14)
    * which in turn builds on $c(a_w^*, w) / c(a_w^*, b)$ which are not yet characterized at this point
    * So let us assume there is a $a_w^* > 0$ somewhere and plot this into the phase diagram
  - Two zero-motion lines for wealth, one in both states
    * The zero-motion line (11.16) for unemployed workers intersects the horizontal axis at $c(a, b) = 0 \Leftrightarrow a_b = -b/r$
    * Note that b/r is the natural borrowing limit
    * The zero-motion line (11.15) for employed workers is by $w - b$ higher than the zero-motion line for unemployed workers

- What the phase diagram also shows us
  - We see four different areas where consumption $c^w(a) = c(a, w)$ of the employed worker and wealth dynamics are illustrated by arrow-pairs
  - We see two different areas for consumption $c^b(a) = c(a, b)$ and wealth of the unemployed worker
  - This gives us an idea how trajectories that satisfy constraints and Keynes-Ramsey rules look like
  - Now is a good time to define equilibrium (our intuition about optimal behaviour)

11.4.2 Equilibrium

- Definition equilibrium
  - Wealth lies in the range $]-b/r, a_w^*[$
  - Consumption and wealth of the employed worker rises on the saddle path $c^w(a)$ and satisfies (11.11)
  - Consumption and wealth of the unemployed worker, $c^b(a)$, satisfy (11.12)
  - Consumption in the (temporary) steady state at a_w^* is by (11.15) given by $c(a_w^*, w) = ra_w^* + w$
  - Relative consumption at a_w^* is given by (11.14) as $\frac{c(a_w^*,w)}{c(a_w^*,b)} = \left(1 - \frac{r-\rho}{s}\right)^{1/\sigma}$
  - [If we knew a_w^*, we could solve the two ODEs with these two boundary conditions]
  - We define a_w^* such that $c^b(-b/r) = 0$ (standard assumption)
  - Alternatively: Fix some exogenous a^{nat} or c^{nat} such that or $c^b(a^{\text{nat}}) = c^{\text{nat}}$

- What next?
  - Solve this numerically
  - Guess an a_w^* and solve ODE system (11.12) and (11.15) with two boundary conditions $c\left(a_w^*, w\right) = ra_w^* + w$ and $c\left(a_w^*, b\right) = \frac{c\left(a_w^*, w\right)}{\left(1 - \frac{r-\rho}{s}\right)^{1/\sigma}}$
  - Check whether $c^b\left(a^{\text{nat}}\right) = c^{\text{nat}}$
    * If yes – done
    * If no – adjust guess for a_w^* (in some systematic way)

11.5 Quantitative findings

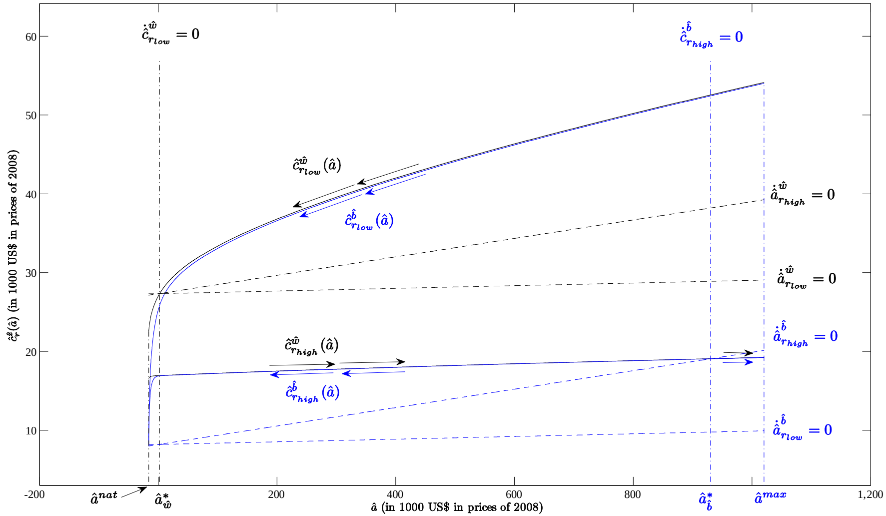


Figure 42 *Policy functions $c(a,w)$ and $c(a,b)$ (for high and low interest rates)*

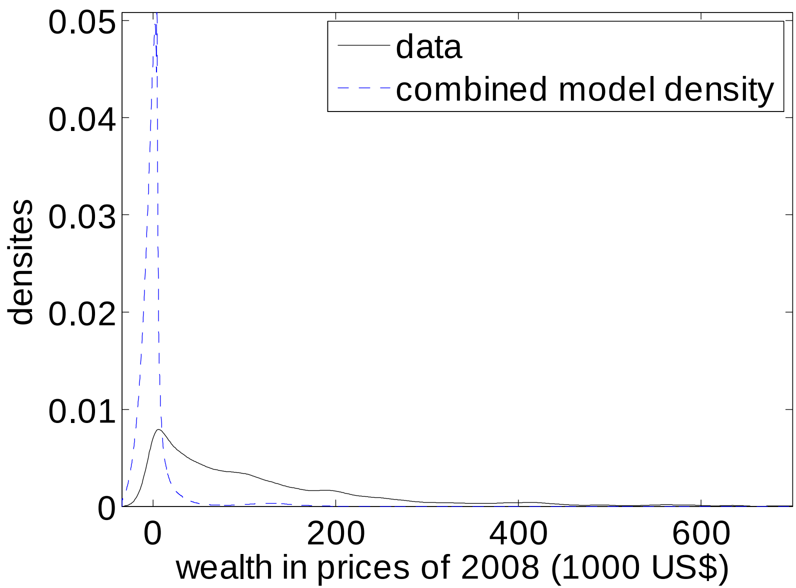


Figure 43 *Wealth distributions in the data and the model after 22 years with labour income risk only (for details see Hoang and Wälde, 2018)*

11.6 What have we learned?

What were the questions again?

- Look at a model which allows to study the effects of unemployment benefits on unemployment when individuals save
- Ask whether such a model can quantitatively replicate observed wealth distributions
- (We abstract from wage distribution here to make central insights clearer)

Findings

- There are too few rich individuals in the model
- Models of this type need an “awesome state” or “superstar state” as in Castañeda et al. (2003) or Kaymak and Poschke (2015)
- See Benhabib and Bisin (2017) and Benhabib, Bisin and Luo (2017) for more background
- Alternative: Allow for capital income risk
  - See Benhabib et al. (2011) for the seminal contribution
  - See Khieu and Wälde (2018) for a theoretical and quantitative analysis of how model distributions fit the evolution of wealth distributions in the NLSY 79 dataset

12 Exercises on unemployment

12.1 Exercises

12.1.1 The matching model

1. Using the setup from the lecture, compute the change in the unemployment rate over time, $\dot{u}(t)$.

2. Find the optimal wage, following Nash bargaining between the firm and the worker, using the following expressions from the lecture,

$$\dot{U}(t) = \rho U(t) - b - a(x)\, p(\theta(t)) \left[E W_y(t) - U(t) \right], \tag{12.1}$$

$$\dot{W}_y(t) = \rho W_y(t) - w_y(t) - \lambda \left[U(t) - W_y(t) \right], \tag{12.2}$$

$$\dot{V}(t) = \rho V(t) + k - a(x)\, q(\theta(t)) \left(E J_y(t) - V(t) \right), \tag{12.3}$$

$$\dot{J}_y(t) = \rho J_y(t) - (y - w_y(t)) - \lambda \left[V(t) - J_y(t) \right]. \tag{12.4}$$

3. Unemployment benefits, given the UI replacement rate ξ, is denoted by

$$b = \xi w_y,$$

with $0 < \xi < 1$. What is the (flow) value of being unemployed, $\rho U(t)$, given this change?

12.1.2 Reservation productivity

1. Using the setup of the lecture, derive the reservation productivity,

$$x = b + \frac{\beta}{1-\beta}\theta k.$$

2. Derive the second differential equation of the classical matching approach to unemployment describing the evolution of labour market tightness, $\dot{\theta}(t)$.
   (Use your answer from 12.1.1 (Question 2) above, as well as the equation for $\dot{J}$.)

3. Given an increase in mean productivity of $h\%$, what is the effect on the reservation wage x?

12.1.3 Pure search model

Using the setup from Cahuc and Zylberberg (2004), determine the reservation wage x, using the value of being employed

$$rV_e = w + s\left(V_u - V_e\right),$$

and unemployed

$$rV_u = z + \lambda \int_x^{\infty} \left[V_e\left(w\right) - V_u\right] h\left(w\right) dw.$$

12.1.4 Non-stationary search

1. What is the value of being employed in the non-stationary search model? Why is it constant?

2. MATLAB programming and non-stationary search

12.1.5 Evolution of the belief

How does the belief, $p(t)$, evolve over time? How can we interpret the result from an economic standpoint?

12.1.6 Optimal precautionary saving

Derive the Keynes-Ramsey rule for optimal precautionary saving.

Part III
Conclusion

13 What did we learn from the individual fields?

13.1 Economic growth

- Central empirical questions
  - Why are some countries rich, why are some others poor?
  - Do countries converge to the same long-run level of income?
  - Is there a reduction of the poverty rate and of inequality as measured by the Gini coefficient?
- Current view of convergence debate
  - Poverty persists but the absolute number declines
  - Inequality as measured by Gini declines as well but very slowly
- Theory of economic growth

- From exogenous factors of growth to endogenous, economically determined drivers of growth
- Policy and politics play a crucial role in shaping the growth path of a society

- Contribution of psychological views
  - Impulse control and savings (Fudenberg and Levine, 2006)
  - Reference points and the impact on optimal growth (Foellmi et al, 2011)
  - Behavioural growth extends existing views on the growth process and allows for novel predictions
  - Empirical relevance still to be seen

13.2 Unemployment

- Central empirical questions
  - How can unemployment meaningfully be defined?
  - How high are unemployment rates in Germany and how do they change over time?
  - How do unemployment rates differ across countries?

- Who is most affected by unemployment? Skill, age, region ...
- How often do individuals become unemployed and how many of them (stocks vs flows)?

- The central theoretical questions
  - Why is there unemployment?
  - How can one reduce unemployment?
  - Can unemployment be reduced without creating poverty?
- Theories of unemployment
  - Traditional theories of labour supply (voluntary unemployment)
  - Traditional theories of real wage rigidities (involuntary unemployment)
  - Pure search views – stresses worker's behaviour
  - Matching models with vacancy creation – stresses the job creation by firms
- Contribution of psychological views
  - time-consistency and time-inconsistent behaviour is widely observed

- this might play a role for theories of unemployment as well
- empirical estimates show that “present bias” is highly relevant for search behaviour of the unemployed

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