Part I
Modelling Money in General Equilibrium: a Primer
Lecture 4
The Basic MIU model: Value function solution

Leopold von Thadden
European Central Bank and University of Mainz

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I Motivation

This lecture goes back to the observation that Walsh (chapter 2) solves the basic MIU model with the value function approach, while we used in Lecture 2, alternatively, the Lagrange multiplier approach.

Against this background, the goal of this lecture is threefold, i.e., we will:

1) give a brief introduction to dynamic programming and the concept of a value function
2) consider a simple and fully tractable example economy and compare the two solution approaches
3) confirm the optimality conditions established in chapter 2 by Walsh
II Value function approach

Basics

- Assume we want to find an infinite sequence of a control variable \( \{c_t\}_{t=0}^{\infty} \) to maximize
  \[
  \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \beta \in (0, 1),
  \]
  subject to the dynamic constraint
  \[
  \omega_{t+1} = g(\omega_t, c_t),
  \]
  where \( \omega \) denotes a state variable with \( \omega_0 \) given

- Let the **value function** \( V(\omega) \) express the **optimal** value of the above problem for any feasible initial value. In particular, define
  \[
  V(\omega_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)
  \]
  where the maximization is subject to \( \omega_{t+1} = g(\omega_t, c_t) \) and \( \omega_0 \) given
II Value function approach

Basics

- Under the **Lagrange multiplier approach** we solved directly for the infinite sequence $\{c_t\}_{t=0}^\infty$

- Alternatively, **dynamic programming** seeks to find a time-invariant policy function $h$ which maps the state $\omega_t$ into the control variable $c_t$. The optimal sequence $\{c_t\}_{t=0}^\infty$ will be indirectly generated by a repeated application of the two functions

\[
\begin{align*}
  c_t &= h(\omega_t) \\
  \omega_{t+1} &= g(\omega_t, c_t),
\end{align*}
\]

ie the policy function and the dynamic constraint, with $\omega_0$ given
II Value function approach
Basics

- Assume we knew $V(\omega)$. Of course, we cannot expect to know $V(\omega)$, since we have not yet solved the problem, but let us proceed on faith.

- If we knew $V(\omega)$ then the policy function $h$ could be computed by solving for each feasible value of $\omega$ the problem

$$\max_c \{ u(c) + \beta V(\tilde{\omega}) \} \quad s.t. \quad \tilde{\omega} = g(\omega, c), \text{ and } \omega \text{ given},$$

exploiting the recursive nature of the original maximization problem (and where $\tilde{\omega}$ denotes the value of $\omega$ in the next period).

- But we don’t know yet the value function $V(\omega)$!

- In other words, rather than to find the infinite sequence $\{c_t\}_{t=0}^{\infty}$ we have transformed the problem such that we need to find the value function $V(\omega)$ and the policy function $c = h(\omega)$ that solve the maximization problem.
II Value function approach

Basics

- The task is to solve jointly for $V(\omega)$ and $h(\omega)$ which are linked by the Bellman equation

$$V(\omega) = \max_c \{ u(c) + \beta V[g(\omega, c)] \}$$

(4)

- The maximizer of the RHS of eqn (4) is a policy function $c = h(\omega)$ that satisfies

$$V(\omega) = u(h(\omega)) + \beta V[g(\omega, h(\omega))]$$

(5)

- Notice that (4) or (5) are functional equations to be solved for the pair of unknown functions $V(\omega)$ and $h(\omega)$
II Value function approach
Features of the solution

- There exist various methods for solving the Bellman equation, depending on the precise nature of the functions $u$ and $g$.
- Under certain assumptions - like the concavity of $u(c)$ and the convexity and compactness of the set $\{(\omega_{t+1}, \omega_t) : \omega_{t+1} \leq g(\omega_t, c_t), c_t \in R\}$ - it turns out that the solution exhibits the following elements:

1) The functional equation (4) has a unique strictly concave solution $V(\bar{\omega})$.

2) This solution is approached in the limit as $j \to \infty$ by iterations on

$$V_{j+1}(\omega) = \max_c \{u(c) + \beta V_j(\bar{\omega})\} \text{ s.t. } \bar{\omega} = g(\omega, c), \text{ and } \omega \text{ given},$$

starting from an initial functional guess $V_0(\bar{\omega})$.

This convergence result leads to a solution procedure which is called **value function iteration**.
II Value function approach

Features of the solution

3) There exists a unique and time invariant optimal policy of the form $c_t = h(\omega_t)$. The derivation of the policy function uses the optimality condition

$$u'(c) + \beta \frac{\partial g(\omega, c)}{\partial c} V'[g(\omega, c)] = 0,$$  \hspace{1cm} (6)

resulting from the maximization of the RHS of (4) w.r.t. $c$

4) The value function $V(\omega)$ is implicitly characterized by

$$V'(\omega) = \beta \frac{\partial g(\omega, h(\omega))}{\partial \omega} V'[g(\omega, h(\omega))] +$$

$$\left\{ u'(h(\omega)) + \beta \frac{\partial g(\omega, h(\omega))}{\partial h} V'[g(\omega, h(\omega))] \right\} \frac{\partial h(\omega)}{\partial \omega},$$

resulting from the maximization of the RHS of (5) w.r.t. $\omega$.

Using (6), this simplifies via the envelope theorem to the expression

$$V'(\omega) = \beta \frac{\partial g(\omega, h(\omega))}{\partial \omega} V'[g(\omega, h(\omega))].$$  \hspace{1cm} (7)
II Value function approach

Comment

- These concepts may seem rather abstract, but they are not when used in practice.

- To see the intuition behind them we will work through a particular example.

- For this example we will verify that the two solution approaches (Value function approach and Lagrange technique) lead to the same results.
III Example: Value function vs. Lagrange approach

- As a particular example of equations (1) and (2), consider the neoclassical growth framework with logarithmic preferences

\[ \sum_{t=0}^{\infty} \beta^t \ln(c_t) \quad \beta \in (0, 1), \]

and Cobb-Douglas production function within the dynamic constraint

\[ k_{t+1} = g(k_t, c_t) = Ak_t^\alpha - c_t, \quad A > 0, \, \alpha \in (0, 1), \]

where \( k \) denotes the single state variable with \( k_0 \) given

- **Comment:**
  - For this example, the timing is chosen to be in line with (1) and (2) such that \( k \) simply replaces \( \omega \)
  - This implies that from the perspective of period \( t \) the variable \( k_t \) is predetermined (rather than \( k_{t-1} \), as assumed by Walsh in chapter 2)
  - For simplicity, we assume \( \delta = 1 \) (ie full depreciation of capital)
III Example: Value function vs. Lagrange approach
Solution via value function iteration

Let us use
\[ \tilde{k} = g(k, c) = Ak^\alpha - c \]

The Bellman equation (5) for the example at hand becomes:
\[
V(k) = \max_c \{ u(c) + \beta V[g(k, c)] \} \quad \text{and } k \text{ given}
\]
\[
= \max_c \{ \ln(c) + \beta V[Ak^\alpha - c] \} \quad \text{and } k \text{ given} \tag{8}
\]

The task is to solve jointly for the value function \( V(k) \) and the policy function \( c = h(k) \) which satisfy
\[
V(k) = u(h(k)) + \beta V[g(k, h(k))]
\]
\[
= \ln(h(k)) + \beta V[Ak^\alpha - h(k)] \tag{9}
\]
To find the pair of functions $V(k)$ and $h(k)$ we employ the procedure of value function iteration:

- for a given initial guess about the value function, called $V_0(k)$, we optimize the RHS of (8) over $c$, establish thereby a policy function $h_1(k)$ and insert it into the RHS of (8) to obtain a new value function $V_1(k)$

- given this new function $V_1(k)$, we optimize, again, over $c$, to obtain a new policy function $h_2(k)$ and a new value function $V_2(k)$

- we iterate on this procedure until convergence has been achieved, i.e., until we have found functions $h_\infty(k) = h(k)$ and $V_\infty(k) = V(k)$ which satisfy (9)
III Example: Value function vs. Lagrange approach
Solution via value function iteration

Value function iteration: **Initial functional guess** \( (j = 0) \)

- Assume
  \[
  V_0(k) = 0
  \]
  This guess holds for all feasible values of \( k \), including \( \tilde{k} \)
- Max of RHS of (8) over \( c \) yields (trivially!)
  \[
  c = h_1(k) = Ak^\alpha \quad \text{and} \quad \tilde{k} = g(k, h_1(k)) = 0
  \]
- Inserting the policy function \( h_1(k) \) into the RHS of (8) leads to
  \[
  V_1(k) = \ln\left(\frac{A}{h_1(k)}\right) + \beta V_0[g(k, h_1(k))] = \ln(A) + \alpha \ln(k)
  \]
III Example: Value function vs. Lagrange approach
Solution via value function iteration

Value function iteration: **Second step** \( (j = 1) \)

* Use the just derived function \( V_1(k) \), ie

\[
V_1(k) = a_1 + \alpha \ln(k)
\]

* Max of RHS of (8) over \( c \) requires

\[
\frac{\partial}{\partial c} [\ln(c) + \beta V_1[Ak^\alpha - c]] = \frac{\partial}{\partial c} [\ln(c) + \beta [a_1 + \alpha \ln(Ak^\alpha - c)]] = 0,
\]

implying

\[
\frac{1}{c} = \alpha \beta \frac{1}{Ak^\alpha - c},
\]

ie we get

\[
c = h_2(k) = \frac{1}{1 + \alpha \beta} Ak^\alpha \quad \text{and} \quad \tilde{k} = g(k, h_2(k)) = \frac{\alpha \beta}{1 + \alpha \beta} Ak^\alpha
\]

* Inserting the policy function \( h_2(k) \) into the RHS of (8) leads to

\[
V_2(k) = \ln\left(\frac{1}{1 + \alpha \beta} Ak^\alpha\right) + \beta \left[ a_1 + \alpha \ln\left(\frac{\alpha \beta}{1 + \alpha \beta} Ak^\alpha\right)\right] = a_2 + (1 + \alpha \beta) \alpha \ln(k)
\]
III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Third step** \( (j = 2) \)

- Use the just derived function \( V_2(k) = a_2 + (1 + \alpha \beta)\alpha \ln(k) \)
- Max of RHS of (8) over \( c \) requires

\[
\frac{\partial}{\partial c} \left[ \ln(c) + \beta V_2[Ak^\alpha - c] \right] = \frac{\partial}{\partial c} \left[ \ln(c) + \beta[a_2 + (1 + \alpha \beta)\alpha \ln(Ak^\alpha - c)] \right] = 0,
\]

implying

\[
\frac{1}{c} = \alpha \beta [1 + \alpha \beta] \frac{1}{Ak^\alpha - c},
\]

ie we get

\[
c = h_3(k) = \frac{1}{1 + \alpha \beta + (\alpha \beta)^2} Ak^\alpha \quad \text{and} \quad \tilde{k} = g(k, h_3(k)) = \frac{\alpha \beta + (\alpha \beta)^2}{1 + \alpha \beta + (\alpha \beta)^2} Ak^\alpha
\]

- Inserting the policy function \( h_3(k) \) into the RHS of (8) leads to

\[
V_3(k) = \ln\left( \frac{Ak^\alpha}{1 + \alpha \beta + (\alpha \beta)^2} \right) + \beta \left[ a_2 + (1 + \alpha \beta)\alpha \ln\left( \frac{\alpha \beta + (\alpha \beta)^2}{1 + \alpha \beta + (\alpha \beta)^2} Ak^\alpha \right) \right]
\]

\[
= a_3 + [1 + \alpha \beta + (\alpha \beta)^2] \alpha \ln(k)
\]
III Example: Value function vs. Lagrange approach
Solution via value function iteration

Value function iteration: **Convergence** \((j \to \infty)\)

- **Policy function:** consider what we got from the iterations done so far:
  
  \[
  h_1(k) = Ak^{\alpha} \\
  h_2(k) = \frac{1}{1 + \alpha\beta} Ak^{\alpha} \\
  h_3(k) = \frac{1}{1 + \alpha\beta + (\alpha\beta)^2} Ak^{\alpha}
  \]

- There is a pattern behind this, i.e., after \(j = T\) steps we will get
  
  \[
  h_T(k) = \frac{1}{1 + \alpha\beta + (\alpha\beta)^2 + (\alpha\beta)^3 + \ldots + (\alpha\beta)^{T-1}} Ak^{\alpha}
  \]

- Recall: \(\alpha \in (0, 1), \beta \in (0, 1)\), implying \(\alpha \cdot \beta \in (0, 1)\)

- Thus, the iteration process ensures that the policy function converges:
  
  \[
  c = h_\infty(k) = h(k) = (1 - \alpha\beta) Ak^{\alpha} \quad (10)
  \]

- Similarly
  
  \[
  \tilde{k} = g(k, h_\infty(k)) = g(k, h(k)) = \alpha\beta Ak^{\alpha} \quad (11)
  \]
Value function iteration: **Convergence** \((j \to \infty)\)

- **Value function**: consider what we got from the iterations done so far:
  \[
  V_0(k) = 0 \\
  V_1(k) = a_1 + \alpha \ln(k) \\
  V_2(k) = a_2 + (1 + \alpha \beta) \alpha \ln(k) \\
  V_3(k) = a_3 + [1 + \alpha \beta + (\alpha \beta)^2] \alpha \ln(k)
  \]

- As concerns the terms including \(\alpha \ln(k)\), there is a pattern behind this, i.e. after \(j = T\) steps we will get
  \[
  V_T(k) = a_T + [1 + \alpha \beta + (\alpha \beta)^2 + (\alpha \beta)^3 + \ldots + (\alpha \beta)^{T-1}] \alpha \ln(k)
  \]

- Since \(\alpha \cdot \beta \in (0, 1)\), for \(j \to \infty\) there will be convergence, i.e.
  \[
  V_\infty(k) = V(k) = a_\infty + \frac{\alpha}{1 - \alpha \beta} \ln(k)
  \]

- What is still missing before we can fully characterize \(V(k)\)?
  \[
  a = a_\infty
  \]
III Example: Value function vs. Lagrange approach

Solution via value function iteration

Value function iteration: **Convergence** \((j \rightarrow \infty)\)

- To find the limit term
  
  \[ a = a_\infty \]

  we could carefully exploit the **algebra of geometric series** to find a converging pattern behind \(a_1, a_2, a_3, \ldots a_T\ldots\)

- Alternatively, we can use the **method of undetermined coefficients**

- Consider eqn (9), ie

  \[
  V(k) = \ln(h(k)) + \beta V[g(k, h(k))],
  \]

  which will be satisfied as \(j \rightarrow \infty\). Combining the so far established limit values we can write this eqn as

  \[
  V(k) = a + \frac{\alpha}{1 - \alpha \beta} \ln(k) = \ln[(1 - \alpha \beta)Ak^\alpha] + \beta[a + \frac{\alpha}{1 - \alpha \beta} \ln(\alpha \beta Ak^\alpha)]
  \]

  \[ h(k) \]

  \[ g(k, h(k)) \]

  - Within this eqn we can determine \(a\) by combining all those terms which are not linked to \(\alpha \ln(k)\)...
III Example: Value function vs. Lagrange approach
Solution via value function iteration

Value function iteration: Convergence \((j \to \infty)\)

...ie from
\[
V(k) = a + \frac{\alpha}{1 - \alpha \beta} \ln(k) = \ln((1 - \alpha \beta)Ak^\alpha) + \beta[a + \frac{\alpha}{1 - \alpha \beta} \ln(\alpha \beta A k^\alpha)]
\]

\(h(k)\)

\(g(k,h(k))\)

we can obtain (via elimination of terms linked to \(\alpha \ln(k)\))
\[
a = \ln((1 - \alpha \beta)A) + a\beta + \frac{\alpha \beta}{1 - \alpha \beta} \ln(\alpha \beta A)
\]
such that the value of \(a\) can be determined as
\[
a = \frac{1}{1 - \beta} \left(\ln((1 - \alpha \beta)A) + \frac{\alpha \beta}{1 - \alpha \beta} \ln(\alpha \beta A)\right)
\]

In sum, using this expression for \(a\), the fully determined value function \(V(k)\) is given by
\[
V(k) = \frac{1}{1 - \beta} \left(\ln((1 - \alpha \beta)A) + \frac{\alpha \beta}{1 - \alpha \beta} \ln(\alpha \beta A)\right) + \frac{\alpha}{1 - \alpha \beta} \ln(k) \tag{12}
\]
III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

- The value function approach is one solution technique among many others
- Alternatively, the problem at hand can be solved with the Lagrange approach
- The value function approach is often used to implicitly characterize optimal solutions of problems for which no explicit solution exists
- Moreover, it is a convenient tool to obtain numerical solutions

→ Since this particular example does have a closed-form solution it is instructive to verify the relationship between the two approaches
III Example: Value function vs. Lagrange approach

Comparison with Lagrange solution

- Consider

\[ \mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \{ \ln(c_t) + \lambda_t [A k_t^\alpha - k_{t+1} - c_t] \} \]

- Optimization of \( \mathcal{L}_t \) over the choice variables \( \{c_t, k_{t+1}, \lambda_t; \forall t \geq 0\} \) leads to a two-dimensional, non-linear system of first-order difference equations in \( c \) and \( k \), i.e.

the consumption Euler equation

\[ \frac{1}{c_t} = \beta^{\alpha} A k_{t+1}^{\alpha-1} \frac{1}{c_{t+1}} \left( f'(k_{t+1}) \right) \]  \hspace{1cm} (13)

and the dynamic resource constraint

\[ k_{t+1} = A k_t^\alpha - c_t \]  \hspace{1cm} (14)

- Moreover, the optimal sequences of variables are subject to the initial condition \( k_0 \) and the terminal condition

\[ \lim_{T \to \infty} \beta^T \lambda_T k_{T+1} = \beta^T \frac{1}{c_T} k_{T+1} = 0 \]
III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

→ To convince ourselves that the two approaches lead to equivalent outcomes we will undertake 3 comparisons:

- Comparison 1: **Transitional dynamics**

- Comparison 2: **Steady-state solution**

- Comparison 3: **Welfare derived from steady-state consumption**
III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

Comparison 1: Transitional dynamics

- Recall from the general set-up discussed above that the value function solution is characterized by eqns of type (6) and (7)
- Eqn (6), ie
  \[ u'(c) + \beta \frac{\partial g(k, c)}{\partial c} V'[g(k, c)] = 0, \]
  results from maximizing the RHS of the Bellman equation w.r.t. to the control variable \( c \). For our example, using the constraint
  \[ \tilde{k} = g(k, c) = Ak^\alpha - c, \]
  it is given by
  \[ \frac{1}{c} = \beta V'({\tilde{k}}) \quad (15) \]
- Eqn (7), ie
  \[ V'(k) = \beta \frac{\partial g(k, h(k))}{\partial k} V'[g(k, h(k))] \]
  implicitly characterizes the optimality of the solution \( V(k) \) via an envelope condition. For our example it is given by
  \[ V'(k) = \beta \alpha Ak^{\alpha-1} V'({\tilde{k}}) \quad (16) \]
III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

- Consider the two eqns (15) and (16), ie

\[
\frac{1}{c} = \beta V'(\tilde{k})
\]

\[
V'(k) = \beta \alpha A k^{\alpha - 1} V'(\tilde{k})
\]

- When forwarded by one period and using

\[
\frac{1}{\tilde{c}} = \beta V'(\tilde{k})
\]

they can be combined to give the consumption Euler equation (13), ie

\[
\frac{1}{c} = \beta \cdot \alpha A(\tilde{k})^{\alpha - 1} \cdot \frac{1}{\tilde{c}} \cdot f'(\tilde{k})
\]

- In sum, the transitional dynamics of the Value function and the Lagrange solutions are characterized by the same difference equations
III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

Comparison 2: Long-run (steady-state) solution

The consumption Euler equation and the dynamic resource constraint derived under the Lagrange approach, ie

$$\frac{1}{c_t} = \beta \alpha A k_{t+1}^{1-\alpha} \frac{1}{f'(k_{t+1}) c_{t+1}} \quad \text{and} \quad k_{t+1} = A k_t^\alpha - c_t$$

are characterized by a unique and saddlepath-stable steady state, with

$$k^* = (\alpha \beta A) \frac{1}{1 - \alpha}$$

$$c^* = A \cdot (k^*)^\alpha - k^* = A \cdot (k^*)^\alpha \cdot [1 - \frac{(k^*)^{1-\alpha}}{A}] = (1 - \alpha \beta) \cdot A \cdot (k^*)^\alpha$$

These steady state values are consistent with eqns (10) and (11) obtained from the value function iteration, ie

$$c = h(k) = (1 - \alpha \beta) A k^\alpha$$

$$\tilde{k} = g(k, h(k)) = \alpha \beta A k^\alpha,$$

where by concavity of $k^\alpha$ the values $\tilde{k}$ and $c$ converge against $k^*$ and $c^*$.
III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

Comparison 3: Welfare derived from steady-state consumption

- Assume the economy is in steady state, ie $k_0 = k^*$
- Then, the welfare of the representative consumer will be given by

$$V(k^*) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t) = \ln(c^*) \cdot \sum_{t=0}^{\infty} \beta^t,$$

amounting to

$$V(k^*) = \frac{1}{1 - \beta} \ln(c^*)$$

$$= \frac{1}{1 - \beta} \left\{ \ln((1 - \alpha \beta)A) + \alpha \ln(k^*) \right\}$$

$$= \frac{1}{1 - \beta} \left\{ \ln((1 - \alpha \beta)A) + \frac{\alpha}{1 - \alpha} \ln(\alpha \beta A) \right\} \quad (17)$$
III Example: Value function vs. Lagrange approach
Comparison with Lagrange solution

Comparison 3: Welfare derived from steady-state consumption

- Recall the value function derived above in the general expression (12), ie

\[
V(k) = \frac{1}{1 - \beta} \left\{ \ln[(1 - \alpha \beta)A] + \frac{\alpha \beta}{1 - \alpha \beta} \ln(\alpha \beta A) \right\} + \frac{\alpha}{1 - \alpha \beta} \ln(k)
\]

- Let \( k = k^* = (\alpha \beta A)^{\frac{1}{1 - \alpha}} \), implying

\[
V(k^*) = \frac{1}{1 - \beta} \ln[(1 - \alpha \beta)A] + \left\{ \frac{1}{1 - \beta} \cdot \frac{\alpha \beta}{1 - \alpha \beta} + \frac{1}{1 - \alpha} \cdot \frac{\alpha}{1 - \alpha \beta} \right\} \ln(\alpha \beta A)
\]

(18)

- Comparing coefficients between \( \ln(\alpha \beta A) \)-related terms, eqns (17) and (18) will be identical if

\[
\frac{1}{1 - \beta} \cdot \frac{\alpha}{1 - \alpha} = \frac{1}{1 - \beta} \cdot \frac{\alpha \beta}{1 - \alpha \beta} + \frac{1}{1 - \alpha} \cdot \frac{\alpha}{1 - \alpha \beta},
\]

which is, indeed, the case

\[ \rightarrow \text{Welfare is identical under value function and Lagrange solutions} \]
Recall from Lecture 2 that we used the Lagrange approach to derive the intertemporal optimality conditions which characterize the basic MIU model.

By contrast, Walsh (Chapter 2) derives these conditions using the value function approach.

In view of the techniques introduced in this lecture it should be no surprise why the two approaches generate identical results.
Recall from Lecture 2 the main ingredients of the intertemporal optimization problem of the representative household.

- Utility function to be maximized via optimal choices of $c_t, m_t, b_t, k_t$:

  $$\sum_{t=0}^{\infty} \beta^t u(c_t, m_t)$$

- Budget constraint (in per capita terms):

  $$f\left(\frac{k_t-1}{1+n}\right) + \tau_t + (1-\delta) \frac{k_{t-1}}{1+n} + \frac{(1+i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + b_t + m_t$$

- Maximization is subject to initial and terminal conditions:
  
  $k_{-1}$ is predetermined and $\lim_{t\to\infty} \beta^t u_{c,t}x_t = 0 \quad x = k, b, m$
To solve this problem via the value function approach it is convenient to introduce a new **state variable** $\omega_t$ which summarizes all **resources** of the representative HH at the beginning of period $t$:

$$\omega_t \equiv f\left(\frac{k_{t-1}}{1+n}\right) + \tau_t + (1 - \delta) \frac{k_{t-1}}{1+n} + \frac{(1 + i_{t-1})b_{t-1} + m_{t-1}}{(1+n)(1+\pi_t)} = c_t + k_t + b_t + m_t$$

Using this definition of $\omega$, we can substitute out for the (old) state variable $k$, ie

$$k_t = \omega_t - c_t - b_t - m_t$$

If one combines the last two eqns, the **law of motion** of the **state variable** $\omega$ can be arranged to satisfy the structure

$$\omega_{t+1} = g(\omega_t, c_t, b_t, m_t), \quad \text{ie}$$

$$\omega_{t+1} = f\left(\frac{\omega_t - c_t - b_t - m_t}{1+n}\right) + \tau_{t+1} + (1 - \delta) \frac{\omega_t - c_t - b_t - m_t}{1+n} + \frac{(1 + i_t)b_t + m_t}{(1+n)(1+\pi_{t+1})} \quad (19)$$
IV The basic MIU model: value function solution

Using (19), the value function satisfies

\[ V(\omega) = \max_{c, b, m} \left\{ u(c, m) + \beta V(\tilde{\omega}) \right\} \]  \hspace{1cm} (20)

subject to

\[ \tilde{\omega} = g(\omega, c, m, b) \]

\[ = f\left( \frac{\omega - c - b - m}{1 + n} \right) + \tilde{\tau} + \frac{1 - \delta}{1 + n} (\omega - c - b - m) + \frac{(1 + i) b + m}{(1 + n)(1 + \tilde{\pi})} \]

and with \( \omega \) given

\[ \text{Notice:} \hspace{0.5cm} \tilde{\tau}, \hspace{0.5cm} \tilde{\pi}, \hspace{0.5cm} \text{and} \hspace{0.5cm} i \hspace{0.5cm} \text{are exogenously given for the representative HH} \]
IV The basic MIU model: value function solution

For suitable assumptions on functional forms (see the corresponding discussion in Lecture 2), eqn (20) is solved by a unique value function \( V(\omega) \), with associated unique policy functions for the control variables \( c \), \( b \), and \( m \).

To characterize the behaviour of the optimal solution, let us derive the optimality conditions

i) for the control variables \( c \), \( b \), and \( m \) in line with eqn (6) and

ii) for the state variable \( \omega \) in line with eqn (7)
IV The basic MIU model: value function solution

Consider the value function (20), i.e.

\[ V(\omega) = \max_{c,b,m} \{ u(c, m) + \beta V(\tilde{\omega}) \} \quad \text{s.t.} \]

\[ \tilde{\omega} = f\left(\frac{\omega - c - b - m}{1 + n}\right) + \tilde{\tau} + \frac{1 - \delta}{1 + n}(\omega - c - b - m) + \frac{(1 + i)b + m}{(1 + n)(1 + \tilde{\pi})} \]

- **Optimal choice of c:**

\[ u_c(c, m) + \beta \frac{\partial \tilde{\omega}}{\partial c} V'(\tilde{\omega}) = 0 \]

\[ u_c(c, m) = \beta \left[ \frac{1}{1 + n} \cdot [f'(k') + 1 - \delta] \right] \cdot V'(\tilde{\omega}) \quad (21) \]

- **Optimal choice of b:**

\[ \beta \frac{\partial \tilde{\omega}}{\partial b} V'(\tilde{\omega}) = 0 \]

\[ \beta \left[ -\frac{1}{1 + n} [f'(k') + 1 - \delta] + \frac{1 + i}{(1 + n)(1 + \tilde{\pi})} \right] \cdot V'(\tilde{\omega}) = 0 \quad (22) \]
IV The basic MIU model: value function solution

Consider the value function (20), i.e.

\[ V(\omega) = \max_{c,b,m} \{ u(c, m) + \beta V(\tilde{\omega}) \} \quad \text{s.t.} \]

\[ \tilde{\omega} = f\left(\frac{\omega - c - b - m}{1 + n}\right) + \tilde{\tau} + \frac{1 - \delta}{1 + n} (\omega - c - b - m) + \frac{(1 + i)b + m}{(1 + n)(1 + \tilde{\tau})} \]

- **Optimal choice of** \( m \):

\[ u_m(c, m) + \beta \frac{\partial \tilde{\omega}}{\partial m} V'(\tilde{\omega}) = 0 \]

\[ u_m(c, m) = \beta \cdot \left[ \frac{1}{1 + n} [f'(k') + 1 - \delta] - \frac{1}{(1 + n)(1 + \tilde{\tau})} \right] \cdot V'(\tilde{\omega}) \quad (23) \]

- **Optimal choice of** \( \omega \):

\[ V'(\omega) = \beta \frac{\partial \tilde{\omega}}{\partial \omega} V'(\tilde{\omega}) \]

\[ V'(\omega) = \beta \left[ \frac{1}{1 + n} [f'(k') + 1 - \delta] \right] \cdot V'(\tilde{\omega}) \quad (24) \]
IV The basic MIU model: value function solution

→ Let us use the definition of the real interest rate

$$1 + r = 1 + f'(k') - \delta$$

and combine the four optimality conditions (21)-(24) to eliminate the terms $V'(\omega)$ and $V'(\tilde{\omega})$:

• Combining (21) and (24), ie

$$u_c(c, m) = \beta \cdot \left( \frac{1 + r}{1 + n} \right) \cdot V'(\tilde{\omega}) \quad \text{and} \quad V'(\omega) = \beta \cdot \left( \frac{1 + r}{1 + n} \right) \cdot V'(\tilde{\omega})$$

gives

$$u_c(c, m) = V'(\omega)$$

and, accordingly,

$$u_c(c, m) = \beta \cdot \left( \frac{1 + r}{1 + n} \right) \cdot u_c(\tilde{c}, \tilde{m}) \quad (25)$$

• Eqn (22) implies

$$1 + r = \frac{1 + i}{1 + \pi} \quad (26)$$
Combining (21) and (23), ie

$$u_c(c, m) = \beta \left[ \frac{1}{1+n} \cdot \left( f'(k') + 1 - \delta \right) \right] \cdot V'(\tilde{\omega})$$

$$u_m(c, m) = \beta \cdot \left[ \frac{1}{1+n} \left[ f'(k') + 1 - \delta \right] - \frac{1}{(1+n)(1+\pi)} \right] \cdot V'(\tilde{\omega})$$

leads to:

$$u_m(c, m) = \beta \cdot \left[ \frac{1+r}{1+n} \cdot V'(\tilde{\omega}) \right] - \beta \frac{1}{(1+n)(1+\pi)} \cdot V'(\tilde{\omega})$$

$$= [1 - \frac{1}{(1+r)(1+\pi)}] \cdot u_c(c, m)$$

$$= \frac{i}{1+i} u_c(c, m)$$

(27)
IV The basic MIU model: value function solution

Summary: Consistent with the system of intertemporal optimality conditions derived in Lecture 2 under the Lagrange approach, eqns (25), (26), and (27) reproduce, respectively:

- **Consumption Euler equation**
  \[
  u_c(c_t, m_t) = \beta \cdot \left( \frac{1 + r_t}{1 + n} \right) \cdot u_c(c_{t+1}, m_{t+1})
  \]

- **Fisher equation**
  \[
  (1 + f'\left(\frac{k_{t-1}}{1 + n}\right) - \delta) \cdot (1 + \pi_{t+1}) = 1 + i_t
  \]
  \[
  \underbrace{\frac{1 + f'\left(\frac{k_{t-1}}{1 + n}\right) - \delta}{1 + r_t}}_{1+r_t}
  \]

- **Optimal allocation rule for real balances**
  \[
  u_m(c_t, m_t) = \frac{i_t}{1 + i_t} u_c(c_t, m_t)
  \]

Moreover, the **resource constraint** closes the system by accounting for the dynamics of the capital stock, ie

\[
  c_t + k_t = f\left(\frac{k_{t-1}}{1 + n}\right) + (1 - \delta) \frac{k_{t-1}}{1 + n}
  \]