Referees' appendix to

The Invariant Distribution of Wealth and Employment Status in a Small Open Economy with Precautionary Savings

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B.1 Derivation of the ODE system (7) for consumption dynamics between jumps

The Bellman equation and first-order condition are taken from (5) and (6) in the main part. Keynes-Ramsey rules are obtained by first computing a differential equation for the evolution of the shadow price of wealth and then re-employing the first-order condition again. For the first step, we need a representation of our two-state Markov chain by a differential equation. It reads

$$dz\left(t\right)=\left\{w-z\left(t\right)\right\}dq_{\lambda}\left(t\right)+\left\{b-z\left(t\right)\right\}dq_{s}\left(t\right) \tag{B.1}$$

where q_λ is a Poisson process with arrival rate λ and q_s is a Poisson process with arrival rate s. In state $z(t)=w$, a jump of q_λ has no effect on $z(t)$, neither does q_s affect $z(t)$ in state $z(t)=b$.

- Evolution of the shadow price

Using the budget constraint (2) and the evolution of labour income (B.1), the differential of the shadow price of wealth reads (suppressing time arguments for simplicity)

$$\begin{aligned} dV_a\left(a,z\right) &= V_{aa}\left(a,z\right)\left\{ra+z-c\right\}dt \\ &\quad+\left[V_a\left(a,w\right)-V_a\left(a,z\right)\right]dq_\lambda+\left[V_a\left(a,b\right)-V_a\left(a,z\right)\right]dq_s. \end{aligned} \tag{B.2}$$

Differentiating the maximized Bellman equation with respect to wealth yields, using the envelope theorem,

$$\begin{aligned} \rho V_a\left(a,z\right) &= \left\{rV_a\left(a,z\right)+\left[ra+z-c\left(a,z\right)\right]V_{aa}\left(a,z\right)\right. \\ &\quad\left.+s\left[V_a\left(a,b\right)-V_a\left(a,z\right)\right]+\lambda\left[V_a\left(a,w\right)-V_a\left(a,z\right)\right]\right\}. \end{aligned} \tag{B.3}$$

Rearranging yields

$$\begin{aligned} &\left(\rho-r\right)V_a\left(a,z\right)-s\left[V_a\left(a,b\right)-V_a\left(a,z\right)\right]-\lambda\left[V_a\left(a,w\right)-V_a\left(a,z\right)\right] \\ &=\left[ra+z-c\left(a,z\right)\right]V_{aa}\left(a,z\right). \end{aligned}$$

Inserting into (B.2) gives

$$\begin{aligned} dV_a\left(a,z\right) &= \left\{\left(\rho-r\right)V_a\left(a,z\right)-s\left[V_a\left(a,b\right)-V_a\left(a,z\right)\right]-\lambda\left[V_a\left(a,w\right)-V_a\left(a,z\right)\right]\right\}dt \\ &\quad+\left[V_a\left(a,w\right)-V_a\left(a,b\right)\right]dq_\lambda+\left[V_a\left(a,b\right)-V_a\left(a,w\right)\right]dq_s. \end{aligned}$$

- Inserting first-order condition

When we now replace the shadow price by marginal utility from the first-order condition (6), we get the Keynes-Ramsey rule for marginal utility,

$$\begin{aligned} du'\left(c\left(a,z\right)\right) = & \left\{\left(\rho - r\right)u'\left(c\left(a,z\right)\right) - s\left[u'\left(c\left(a,b\right)\right) - u'\left(c\left(a,z\right)\right)\right]\right. \\ & \left. -\lambda\left[u'\left(c\left(a,w\right)\right) - u'\left(c\left(a,z\right)\right)\right]\right\} dt \\ & + \left[u'\left(c\left(a,w\right)\right) - u'\left(c\left(a,z\right)\right)\right] dq_\lambda + \left[u'\left(c\left(a,b\right)\right) - u'\left(c\left(a,z\right)\right)\right] dq_s. \end{aligned} \tag{B.4}$$

- Keynes-Ramsey rules for specific states

This equation is the Keynes-Ramsey rule for both states. To make it more informative, we now derive one Keynes-Ramsey rule for each state. For an employed individual, the state and labour income is given by $z = w$. For reasons that become clear in Section A.1 below, denote wealth in the state by $a = a_w$. The rule in (B.4) then reads

$$\begin{aligned} du'\left(c\left(a_w,w\right)\right) = & \left\{\left(\rho - r\right)u'\left(c\left(a_w,w\right)\right) - s\left[u'\left(c\left(a_w,b\right)\right) - u'\left(c\left(a_w,w\right)\right)\right]\right\} dt \\ & + \left[u'\left(c\left(a_w,b\right)\right) - u'\left(c\left(a_w,w\right)\right)\right] dq_s. \end{aligned}$$

Let $f\left(.\right)$ be the inverse function for u', i.e. $f\left(u'\right) = c$ and compute the differential of $f\left(u'\left(c\left(a_w,w\right)\right)\right)$. This gives

$$\begin{aligned} df\left(u'\left(c\left(a_w,w\right)\right)\right) = & f'\left(u'\left(c\left(a_w,w\right)\right)\right)\left\{\left(\rho - r\right)u'\left(c\left(a_w,w\right)\right) - s\left[u'\left(c\left(a_w,b\right)\right) - u'\left(c\left(a_w,w\right)\right)\right]\right\} dt \\ & + \left[f\left(u'\left(c\left(a_w,b\right)\right)\right) - f\left(u'\left(c\left(a_w,w\right)\right)\right)\right] dq_s. \end{aligned}$$

As $f\left(u'\right) = c$ and therefore $f'\left(u'\left(c\left(a_w,w\right)\right)\right) = \frac{df(u'(c(a_w,w)))}{du'(c(a_w,w))} = \frac{dc(a_w,w)}{du'(c(a_w,w))} = \frac{1}{u''(c(a_w,w))}$, we get

$$\begin{aligned} dc\left(a_w,w\right) = & \frac{1}{u''\left(c\left(a_w,w\right)\right)}\left\{\left(\rho - r\right)u'\left(c\left(a_w,w\right)\right) - s\left[u'\left(c\left(a_w,b\right)\right) - u'\left(c\left(a_w,w\right)\right)\right]\right\} dt \\ & + \left[c\left(a_w,b\right) - c\left(a_w,w\right)\right] dq_s \Leftrightarrow \\ \frac{u''\left(c\left(a_w,w\right)\right)}{u'\left(c\left(a_w,w\right)\right)} dc\left(a_w,w\right) = & \left\{\rho - r - s\left[\frac{u'\left(c\left(a_w,b\right)\right)}{u'\left(c\left(a_w,w\right)\right)} - 1\right]\right\} dt \\ & + \frac{u''\left(c\left(a_w,w\right)\right)}{u'\left(c\left(a_w,w\right)\right)}\left[c\left(a_w,b\right) - c\left(a_w,w\right)\right] dq_s. \end{aligned}$$

Multiplying by -1 yields

$$\begin{aligned} -\frac{u''\left(c\left(a_w,w\right)\right)}{u'\left(c\left(a_w,w\right)\right)} dc\left(a_w,w\right) = & \left\{r - \rho + s\left[\frac{u'\left(c\left(a_w,b\right)\right)}{u'\left(c\left(a_w,w\right)\right)} - 1\right]\right\} dt \\ & - \frac{u''\left(c\left(a_w,w\right)\right)}{u'\left(c\left(a_w,w\right)\right)}\left[c\left(a_w,b\right) - c\left(a_w,w\right)\right] dq_s. \end{aligned} \tag{B.5}$$

Using the instantaneous CRRA utility function (1), we get $\frac{u''(c(a_w,w))}{u'(c(a_w,w))} = \frac{-\sigma c(a_w,w)^{-\sigma-1}}{c(a_w,w)^{-\sigma}} = \frac{-\sigma}{c(a_w,w)}$ and therefore

$$\sigma \frac{dc\left(a_w,w\right)}{c\left(a_w,w\right)} = \left\{r - \rho + s\left[\left(\frac{c\left(a_w,w\right)}{c\left(a_w,b\right)}\right)^\sigma - 1\right]\right\} dt + \sigma\left[\frac{c\left(a_w,b\right)}{c\left(a_w,w\right)} - 1\right] dq_s.$$

After dividing by σ, we get (A.1a) in the next section. The derivation of (A.1c) also starts from (B.4) and steps are in perfect analogy.