

Johannes Gutenberg-Universität Mainz
Master in International Economics and Public Policy

Chair in Macroeconomics

Economic Methods

2019/20 winter term

Jean Roch Donsimoni

Contents

1	Introduction	1.0
2	Preliminaries: formulating a problem in economics	2.0
I	Static Utility Maximisation	3.0
3	Typical static maximisation problem	3.0
3.1	Setup	3.0
3.2	Example: Optimal goods allocation (Prof. Schunk: PPE)	3.2
3.3	Exercises	3.5
II	Cost Minimisation	4.0
4	Typical minimisation problem	4.0
4.1	Setup	4.0
4.2	Example: firm cost minimisation (Prof. Harms: Int'l Trade)	4.2
4.3	Exercises	4.5

III	Profit Maximisation	5.0
5	Profit maximisation and factor demands	5.0
5.1	Setup	5.0
5.2	Example: Capital demand (Prof. Barbaro: Devp't + Growth)	5.2
5.3	Exercises	5.4
6	Profit maximisation under monopoly	6.0
6.1	Setup	6.0
6.2	Example: Lobbying and rent creation (Prof. Schunk: PPE)	6.2
6.3	Exercises	6.6
IV	Envelope Theorem	7.0
7	Definition and application	7.0
7.1	Preliminaries	7.0
7.2	Example: Shephard's Lemma (Prof. Harms: Int'l Trade)	7.2
7.3	Exercises	7.4
V	Intertemporal Maximisation Problem	8.0

8	Discrete time model	8.0
8.1	Setup	8.0
8.2	Example: Sovereign default (Prof. Sauré: Fin. Economics I)	8.2
8.3	Exercises	8.3
9	Continuous time problem	9.0
9.1	Setup	9.0
9.2	Example: the central planner (Prof. Wälde: Advanced Macro)	9.2
9.3	Exercises	9.4
VI	Ordinary Least Squares	10.0
10	Typical OLS regression	10.0
10.1	Preliminaries: method of moments for 1 explanatory variable	10.0
10.2	Example: Multiple Regression (Prof. van Ewijk: Intro Econometrics)	10.3
10.3	Exercises	10.4
VII	Probability Theory	11.0
11	Distribution	11.0

11.1 Discrete random variable 11.0

11.2 Continuous random variable 11.1

11.3 Example: Distribution of discrete and continuous RV (Prof. Schank: Stats
Methods) 11.2

11.4 Exercises 11.5

12 Mean and variance **12.0**

12.1 Discrete random variable 12.0

12.2 Continuous random variable 12.1

12.3 Example: Mean and Variance of discrete and continuous RV (Prof. Schank:
Stats Methods) 12.2

12.4 Exercises 12.3



GUTENBERG SCHOOL OF
MANAGEMENT
& ECONOMICS



Johannes Gutenberg-Universität Mainz
Master in International Economics and Public Policy

Chair in Macroeconomics

Economic Methods

2019/20 winter term

Jean Roch Donsimoni

Contact info:

Office: 01-126 (old ReWi)

E-mail: jdonsimo@uni-mainz.de

Office hours: by appointment only

1 Introduction

This *Economic Methods* course

- covers
 - Static utility maximisation
 - Cost minimisation
 - Profit maximisation
 - * Profit maximisation and factor demands
 - * Profit maximisation under monopoly
 - Envelope theorem
 - Intertemporal maximisation
 - Basic probability theory
 - Ordinary least squares
- lasts 7 weeks and ends before Christmas break
- has no formal examination at the end

- is intended for
 - students who have not done economic theory before (or a long time ago)
 - students who need a little more help connecting abstract maths concepts to economic problems
- builds on
 - *Maths Review* (by Alex Dzionara): covers basic linear algebra, real analysis, and some probability theory
 - *MIEPP courses* (compulsory and elective): pose economic problems to be solved
 - \Rightarrow *this course helps connect the two*

Therefore we begin after everyone else ...

Table 1: MIEPP Timetable Winter Semester 2019/2020

Week #	Dates	Courses
1	14/10 to 18/10	<i>“Maths Review”</i>
2	21/10 to 25/10	<i>“Core Modules”</i> lectures start
3	28/10 to 1/11	<i>“Economic Methods”</i> starts
...
9	9/12 to 13/12	<i>“Economic Methods”</i> ends
...

The course format will be structured as follows...

- Classes twice a week: 1 lecture + 1 tutorial (see [Macroeconomics Chair website](#) for details)
- Lecture (90 mins):
 - (i) We look at a typical economic problem (e.g. utility maximisation).
 - (ii) A solution method is introduced for this class of problems (e.g. Lagrangian).
 - (iii) We discuss and explain an example from one of the other lectures (e.g. PPE).
- Tutorial (90 mins):
 - (i) You (the students) solve problems on the board in class.
 - (ii) At the end, we check your steps/results together.

⇒ Course material available online on the [Macroeconomics Chair website](#) every Friday for the following week.

Table 2: Mapping of topics with MIEPP courses

Part	Topic	Professor (<i>course</i>)	Semester	Section
I	Static maximisation	Schunk (<i>Principles Pub Econ</i>)	Winter	3
II	Cost minimisation	Harms (<i>International Trade</i>)	Winter	4
III	Profit max.: - <i>factor demand</i> - <i>monopoly power</i>	Barbaro (<i>Dev't + Growth</i>)	Winter	5
		Schunk (<i>Principles Pub Econ</i>)	Winter	6
IV	Envelope theorem	Harms (<i>International Trade</i>)	Winter	7
V	Intertemp. max.: - <i>discrete time</i> - <i>continuous time</i>	Sauré (<i>Fin Econonometrics I</i>)	Summer	8
		Wälde (<i>Advanced Macro</i>)	Winter	9
VI	Ordinary least squares	van Ewijk (<i>Intro Econometrics</i>)	Winter	10
VII	Probability: - <i>distribution</i> - <i>mean and variance</i>	Schank (<i>Stats Methods</i>)	Summer	11
		Schank (<i>Stats Methods</i>)	Summer	12

2 Preliminaries: formulating a problem in economics

Simplest possible definition:

Economic theory studies the optimal behaviour of agents.

How do we do this? We define

- an **objective function** (or payoff in game theory), i.e. the goal the agent is trying to achieve,
- a set of possible *choices*,
- and a **trade-off**, i.e. some cost to the choices made.

⇒ no trade-off = no problem!

Next, we apply mathematical tools to ...

- ... find the best possible *choice* given the **objective** and the **trade-off** faced by the agent.

Why use maths?

- Forces us to have clear definitions.
- Helps to have clear relationships between various elements of the model.
- Helps to apply logic to find the “true” best possible solution to a problem.
 - \Rightarrow 1 economic model tackles 1 problem, simplified to its bare minimum.

Empirical economics uses statistics (a sub-branch of mathematics) to

- find out if a model accurately represents reality (i.e. data),
- and test predictions of models.
- (*also to test relationships between variables without model \rightsquigarrow bad research*)

\Rightarrow holds economic theory accountable, while economic theory gives problems for empirical economics to investigate, i.e. they complement each other.

Part I

Static Utility Maximisation

3 Typical static maximisation problem

3.1 Setup

Consider the following consumption allocation problem,

$$\max_{\{c_i\}_{i=1}^N} u(c_1, \dots, c_N)$$

subject to

$$\sum_{i=1}^N p_i c_i = W$$

- An individual makes *choices* over many consumption goods ($\{c_i\}_{i=1}^N := c_1, \dots, c_N$) given a budget set ($W > 0$) to maximise *utility* at a given point in time (micro perspective \rightsquigarrow static). Each good i costs p_i .
- The problem can be solved using the ***Lagrangian*** method

- Lagrangian method consists in (i) setting up the Lagrangian function and (ii) taking the first-order conditions (FOCs) to determine the optimal solution.

(i) First we set up the Lagrangian function

$$\mathbb{L}(c_1, \dots, c_N) = u(c_1, \dots, c_N) + \lambda \left[W - \sum_{i=1}^N p_i c_i \right],$$

(ii) next we take the FOCs

$$\frac{\partial \mathbb{L}}{\partial c_i} = \frac{\partial u(c_1, \dots, c_N)}{\partial c_i} - \lambda p_i = 0, \forall i.$$

- From this, we can also compute the marginal rate of substitution (MRS) between two goods $i, j \in [1, N]$ defined as

$$MRS_{ij} = \frac{\partial u(c_1, \dots, c_N) / dc_i}{\partial u(c_1, \dots, c_N) / dc_j}$$

\Rightarrow useful for computing slope indifference curves ($= -MRS$)

3.2 Example: Optimal goods allocation (Prof. Schunk: PPE)

Optimal allocation of goods

- Formally:
Maximize the utility of A for a given utility of B and the resource constraint:

$$\begin{aligned} \max_{x_1^A, x_2^A, x_1^B, x_2^B} u(x_1^A, x_2^A) \quad & s.t. \quad u(x_1^B, x_2^B) \geq \bar{u} \\ & x_1^A + x_1^B = e_1^A + e_1^B \\ & x_2^A + x_2^B = e_2^A + e_2^B \end{aligned}$$

- Lagrange function:

$$\begin{aligned} L = & u(x_1^A, x_2^A) + \lambda(u(x_1^B, x_2^B) - \bar{u}) + \mu_1(x_1^A + x_1^B - e_1^A - e_1^B) \\ & + \mu_2(x_2^A + x_2^B - e_2^A - e_2^B) \end{aligned}$$

Optimal allocation of goods

- Lagrange function:

$$L = u(x_1^A, x_2^A) + \lambda(u(x_1^B, x_2^B) - \bar{u}) + \mu_1(x_1^A + x_1^B - e_1^A - e_1^B) + \mu_2(x_2^A + x_2^B - e_2^A - e_2^B)$$

- First order conditions:

$$x_1^A : \frac{\partial u^A}{\partial x_1^A} + \mu_1 = 0$$

$$x_2^A : \frac{\partial u^A}{\partial x_2^A} + \mu_2 = 0$$

$$x_1^B : \lambda \frac{\partial u^B}{\partial x_1^B} + \mu_1 = 0$$

$$x_2^B : \lambda \frac{\partial u^B}{\partial x_2^B} + \mu_2 = 0$$

Optimal allocation of goods

- From the FOCs, it follows:

$$\frac{\mu_1}{\mu_2} = \frac{\frac{\partial u^A}{\partial x_1^A}}{\frac{\partial u^A}{\partial x_2^A}} = \frac{\frac{\partial u^B}{\partial x_1^B}}{\frac{\partial u^B}{\partial x_2^B}}$$

$$\boxed{-MRS^A = -MRS^B}$$

- Interpretation:
An efficient allocation is achieved when the marginal rates of substitution (in consumption) are equal for all individuals
- See also: Contract curve

3.3 Exercises

In a world with two goods, 1 and 2, and two individuals A and B , what is the rule governing the optimal allocation of goods between A and B (i.e. the MRS_{AB})? Use the same setup as in Example 3.2.

1. Use the following functional form: $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$, with $0 < \alpha < 1$.
2. Use the following functional form: $u(x_1, x_2) = \ln x_1 + \ln x_2$.
3. Use the following functional form: $u(x_1, x_2) = \frac{x_1^{1-\sigma}-1}{1-\sigma} + \frac{x_2^{1-\sigma}-1}{1-\sigma}$, with $\sigma > 0$ and $\sigma \neq 1$.
4. Use the following functional form: $u(x_1, x_2) = \sum_{i=1}^2 x_i^\theta$, with $\theta > 0$.

Part II

Cost Minimisation

4 Typical minimisation problem

4.1 Setup

Consider the following problem where costs of production are a function of inputs and prices

$$\min_{\{v_i\}_{i=1}^N} c(v_1, \dots, v_N) = \sum_{i=1}^N v_i w_i$$

The firm uses factor inputs $\{v_i\}_{i=1}^N := v_1, \dots, v_N$ to produce a given level of output y using a technology $f(\cdot)$, where $f_{v_i}(\cdot) := \partial f(\cdot)/\partial v_i > 0$ and $f_{v_i v_i}(\cdot) := \partial^2 f(\cdot)/\partial v_i^2 < 0$,

$$y = f(v_1, \dots, v_N),$$

- For a given output level y , a firm chooses a combination of factor inputs v_1, \dots, v_N to minimise the costs at given factor prices $\{w_i\}_{i=1}^N$.
- Solving this problem yields *conditional factor demands*

- Once more, we can use the Lagrangian method,

$$\mathbb{L}(v_1, \dots, v_N) = \underbrace{\sum_{i=1}^N v_i w_i}_{=c(v_1, \dots, v_N)} + \lambda [y - f(v_1, \dots, v_N)],$$

and once again obtain the FOCs

$$\frac{\partial \mathbb{L}}{\partial v_i} = w_i - \lambda \frac{\partial f(v_1, \dots, v_N)}{\partial v_i} = 0, \forall i$$

such that looking at two separate factors of production i and j , where $i \neq j$, we would have

$$w_i = \lambda \frac{\partial f(v_1, \dots, v_N)}{\partial v_i},$$

$$w_j = \lambda \frac{\partial f(v_1, \dots, v_N)}{\partial v_j},$$

and dividing one by the other yields the optimality condition

$$\frac{w_i}{w_j} = \frac{\partial f(v_1, \dots, v_N) / \partial v_i}{\partial f(v_1, \dots, v_N) / \partial v_j}.$$

4.2 Example: firm cost minimisation (Prof. Harms: Int'l Trade)

Harms/International Trade



Supply

- The firm's cost minimization problem (contd.):
 - Special case: Two factors of production (K and L) with factor prices r (rental rate of capital) and w (wage).

$$\underset{K_i, L_i}{Min} \quad c_i = r K_i + w L_i$$

subject to ("s.t.") the constraint: $f_i(K_i, L_i) = y_i$

- **Iso-cost line**: $K_i = \frac{c_i}{r} - \frac{w}{r} L_i$ (slope: - w/r)

- Optimality condition:

$$\frac{w}{r} = \frac{\partial f_i(K_i, L_i) / \partial L_i}{\partial f_i(K_i, L_i) / \partial K_i}, \text{ i.e. optimum defined by } \textbf{point of tangency} \\ \text{between isoquant and iso-cost line.}$$

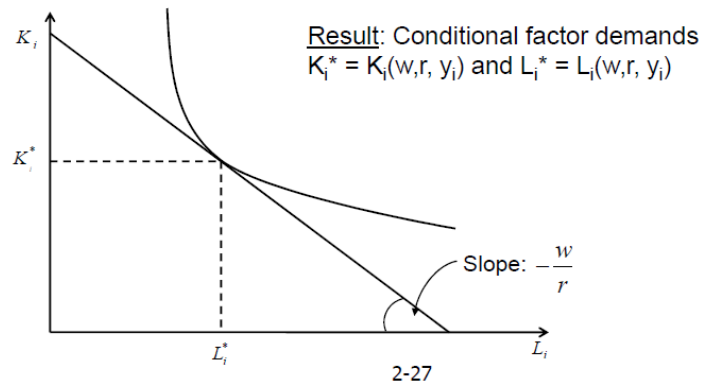
2-26

Harms/International Trade

Supply

- The firm's cost minimization problem (contd.):

Figure 2.4: The firm's cost minimization problem with 2 factor inputs



The problem above gives us the following Lagrangian function

$$\mathbb{L}(K_i, L_i) = rK_i + wL_i + \lambda[y_i - f_i(K_i, L_i)],$$

with FOCs

$$\begin{aligned}\frac{\partial \mathbb{L}}{\partial K_i} &= r - \lambda \frac{\partial f_i(K_i, L_i)}{\partial K_i} = 0 \Leftrightarrow r = \lambda \frac{\partial f_i(K_i, L_i)}{\partial K_i} \\ \frac{\partial \mathbb{L}}{\partial L_i} &= w - \lambda \frac{\partial f_i(K_i, L_i)}{\partial L_i} = 0 \Leftrightarrow w = \lambda \frac{\partial f_i(K_i, L_i)}{\partial L_i}.\end{aligned}$$

Dividing the second FOC by the first one we obtain the optimality condition from the slides above

$$\frac{w}{r} = \frac{\partial f_i(K_i, L_i)/\partial L_i}{\partial f_i(K_i, L_i)/\partial K_i}.$$

4.3 Exercises

Consider a firm i producing output $y_i \geq 0$ using technology $f_i(K_i, L_i) \geq 0$ using two factors of production, $K_i \geq 0$ and $L_i \geq 0$. What is the optimality condition governing factors demand? Use the same setup as in Example 4.2.

1. Use the following functional form: $f_i(K_i, L_i) = K_i^\alpha L_i^{1-\alpha}$, with $0 < \alpha < 1$.
2. Use the following functional form: $f_i(K_i, L_i) = [\alpha K_i^\epsilon + (1 - \alpha)L_i^\epsilon]^{\frac{1}{\epsilon}}$, with $0 < \alpha < 1$ and $\epsilon < 1$.

Part III

Profit Maximisation

5 Profit maximisation and factor demands

5.1 Setup

Consider the following problem where a firm tries to maximise profits π

$$\max_{\{v_i\}_{i=1}^N} \pi = y - c(v_1, \dots, v_N)$$

subject to

$$y = f(v_1, \dots, v_N)$$

and

$$c(v_1, \dots, v_N) = \sum_{i=1}^N w_i v_i$$

- Given factor prices $\{w_i\}_{i=1}^N$, a firm chooses a combination of factor inputs $\{v_i\}_{i=1}^N$ to maximise its profits

- Solving this problem yields *unconditional factor demands*
- We can solve this problem by the Lagrangian method once more,

$$\mathbb{L}(v_1, \dots, v_N) = y - c(v_1, \dots, v_N) + \lambda[f(v_1, \dots, v_N) - y] + \mu \left[c(v_1, \dots, v_N) - \sum_{i=1}^N w_i v_i \right]$$

... but this becomes a little complicated.

- Instead, we can employ a simpler method, known as the substitution method, where (i) we replace terms by their definitions and (ii) solve directly for optimal factor allocation:

$$\max_{\{v_i\}_{i=1}^N} \pi = y - c(v_1, \dots, v_N) \Leftrightarrow \max_{\{v_i\}_{i=1}^N} \pi = f(v_1, \dots, v_N) - \sum_{i=1}^N w_i v_i,$$

giving us the FOCs,

$$\frac{\partial \pi}{\partial v_i} = \frac{\partial f(v_1, \dots, v_N)}{\partial v_i} - w_i = 0 \Leftrightarrow w_i = \frac{\partial f(v_1, \dots, v_N)}{\partial v_i},$$

such that each factor of production is used up until its marginal effect on production equals its price (i.e. its marginal cost).

5.2 Example: Capital demand (Prof. Barbaro: Devp't + Growth)

3. Openness and Factor Accumulation

- Growth with Capital Mobility
 - Production function: $y = Ak^\alpha$
 - Marginal Product of Capital: $MPK = \alpha Ak^{\alpha-1}$
 - Rental rate of capital: $r = MPK = \alpha Ak^{\alpha-1}$
 - Perfect openness to capital flows: $r = r_w$
 - Level of capital per worker: $r_w = \alpha Ak^{\alpha-1} \Leftrightarrow k = (\alpha A / r_w)^{1/(1-\alpha)}$
 - Result:
 - If $r_{\text{home}} < r_w$: capital mobility leads to a lower capital stock
 - If $r_{\text{home}} > r_w$: capital mobility leads to an increasing capital stock
- If the country is small with perfect capital mobility, domestic savings are negligible for the growth rate of a country

This model starts from a representative firm producing aggregate output Y using aggregate capital K and labour L with a Cobb-Douglas production function with parameter $0 < \alpha < 1$ and total factor productivity (TFP) A

$$Y = AK^\alpha L^{1-\alpha}.$$

Dividing through by L yields output per worker $y = Ak^\alpha$. The representative firm maximises profits, which we can also normalise by L

$$\pi = Y - rK - wL \Leftrightarrow \hat{\pi} := \frac{\pi}{L} = y - rk - w.$$

Maximising profits per worker by choosing capital per worker optimally, using the substitution method, we obtain

$$\begin{aligned} \frac{\partial}{\partial k} \hat{\pi} &= \frac{\partial}{\partial k} (Ak^\alpha - rk - w) = 0 \\ \Leftrightarrow r &= \alpha Ak^{\alpha-1}. \end{aligned}$$

If r is fixed because the country is small in comparison to the world (so is a price taker for capital, which is mobile, and $r = r_w$), we can rearrange the condition above for the optimal amount of capital per worker

$$k = \left(\frac{\alpha A}{r_w} \right)^{\frac{1}{1-\alpha}}.$$

5.3 Exercises

Consider a country producing aggregate output $Y \geq 0$ using technology $F(K, L) \geq 0$ using aggregate capital, $K \geq 0$ and aggregate labour $L \geq 0$. What will be capital per worker be equal to? Use the same setup as in Example 4.2.

1. Use the following functional form: $Y = K^\alpha (AL)^{1-\alpha}$, with $0 < \alpha < 1$.
2. Use the following functional form: $Y = [\alpha K^\theta + (1 - \alpha)(AL)^\theta]^{1/\theta}$, with $0 < \alpha < 1$ and $\theta < 1$.

6 Profit maximisation under monopoly

6.1 Setup

Consider the following profit maximisation problem for some firm with monopoly power, where we now consider that the price in this market will depend on the output of the firm (*was not the case before with perfect competition, because each firm was too small to affect the price* → *not the case with monopoly*)

$$\max_y \pi = p(y)y - c(w_1, \dots, w_N, y)$$

- For a given cost function $c(w_1, \dots, w_N, y)$, the monopolistic firm chooses output y in order to maximise profits, thus setting optimal output production.
- Solving this problem yields the optimal monopolistic price as the reciprocal of the optimal output: $y(p) \Leftrightarrow p := y^{-1}(p)$. In other words: choosing output fixes the price in the market, because the firm has “monopoly power”.

- To solve this problem, we can immediately take the derivative w.r.t. output y and solve for the optimal level...

$$\begin{aligned}\frac{\partial \pi}{\partial y} &= p'(y)y + p(y) - c_y(w_1, \dots, w_N, y) = 0 \\ \Leftrightarrow c_y(w_1, \dots, w_N, y) &= p'(y)y + p(y) \\ c_y(w_1, \dots, w_N, y) &= p(y) \left[\underbrace{\frac{dp(y)}{dy} \frac{y}{p(y)}}_{\text{= elasticity of price in output}} + 1 \right]\end{aligned}$$

- Rent creation is one possible application of monopoly power, whereby a firm creates a monopoly in order to extract “rent” from consumers.
 \Rightarrow solve for $y = c_y^{-1}(p)$, i.e. apply the inverse of the derivative of the cost function to the price to obtain optimal output (= rearrange)

6.2 Example: Lobbying and rent creation (Prof. Schunk: PPE)

Government Policy - Lobbying

A lobbying model: Assumptions

- Consider a small open economy in which two consumption goods are produced.
 - Small economy: Treat prices as fixed.
- Without tariffs, the world prices of these two commodities is 1.
- Some output is consumed and some is exported.
- A quantity l^* of labor is supplied inelastically by consumers, and it is divided between production of the 2 goods and lobbying.
- Good 2 is produced with constant returns to scale, and one unit of labor produces one unit of output
→ wage rate w must equal 1 (higher→loss; lower→unlimited profit, bc. prices fixed)
- Cost function for the firm producing good 1:
 $C(y_1) = \frac{1}{2} y_1^2$, where y_1 is output.
- With a tariff t (which may be zero), the price of good 1 on the domestic market becomes $1 + t$.

Government Policy - Lobbying

A lobbying model:

- Assuming that all of the output of the firm is sold on the domestic market, the profit level of the firm is:
$$\pi_1(t) = y_1[1+t] - \frac{1}{2} y_1^2$$
- Profit is maximized at output level:
$$y_1 = 1 + t$$

Output of good 1 is increasing in the value of the tariff
- Level of profit resulting from this output:
$$\pi_1(t) = \frac{1}{2} [1+t]^2 \quad (\text{Profit increases as the square of the tariff!})$$
- Equilibrium on the labor market: $I^* = I_1 + I_2 + I_L$. (I_L : labor for lobbying)
 - $I_1 = \frac{1}{2} [1+t]^2$
 - I_L : Is determined using Complete Dissipation Theorem:
Resources are used for lobbying up to the point at which the extra profit they generate equals the resource cost

Government Policy - Lobbying

- Without lobbying, profit is $\pi_1(0)$. After successful lobbying, it is $\pi_1(t)$.
- The value of labor that the firm devotes to lobbying is therefore:
- $l_L = \pi_1(t) - \pi_1(0) = \frac{1}{2} [1+t]^2 - \frac{1}{2} = \frac{1}{2} [2t+t^2]$
 - Hence, the value of labor wasted in lobbying is increasing as the square of the tariff
- Since production of each unit of good 2 requires 1 unit of labor: $l_2 = y_2$
- Hence, $l_2 = l^* - l_1 - l_L$. Or: $y_2 = l^* - \frac{1}{2} [1+4t+2t^2]$
 - Output of good 2 is decreasing as the square of the tariff.
 - The production of good 2 falls at a faster rate than the production of good 1 increases
- To quantify the effect, look at value of national output at world prices:
 $y_1 + y_2 = l^* - \frac{1}{2} [2t^2+2t-1]$
 - National income is reduced at the rate of the tariff squared.
- **Rent-seeking can be very damaging.**

In this framework the firm chooses y_1 optimally, maximising profits π_1 , thus

$$\frac{\partial \pi_1}{\partial y_1} = (1 + t) - y_1 = 0 \Leftrightarrow y_1 = 1 + t.$$

Replacing in the profit function yields

$$\begin{aligned}\pi_1 &= y_1(1 + t) - \frac{1}{2}y_1^2 \\ &= (1 + t)^2 - \frac{1}{2}(1 + t)^2 \\ &= \frac{1}{2}(1 + t)^2.\end{aligned}$$

6.3 Exercises

Consider the lobbying model from Example 6.2, and solve the problem faced by the producer of good 1, using a cost function that is

1. cubic, i.e. $C(y_1) = \frac{1}{3}y_1^3$. What is the optimal level of output y_1 ? What are profits π_1 given by at this production level?
2. exponential, i.e. $C(y_1) = e^{y_1}$. What is the optimal level of output y_1 ? What are profits π_1 given by at this production level?

Part IV

Envelope Theorem

7 Definition and application

7.1 Preliminaries

Theorem Let $f(x_1, \dots, x_N; a)$ be a continuous function of $(x_1, \dots, x_N) \in \mathbf{R}^n$ and the scalar a . For each choice of parameter a , consider the unconstrained maximisation problem

$$\max_{\{x_i\}_{i=1}^N} f(x_1, \dots, x_N; a).$$

Let $x_1^*(a), \dots, x_N^*(a)$ be solutions to this problem. Suppose that $x_1^*(a), \dots, x_N^*(a)$ are continuous functions of a . Then,

$$\frac{d}{da} f(x_1^*(a), \dots, x_N^*(a); a) = \frac{\partial}{\partial a} f(x_1^*(a), \dots, x_N^*(a); a).$$

In other words, $\frac{dx_i^*(a)}{da} = 0$ and we only need to look at the partial derivative of $f(\cdot)$ w.r.t. a as a direct argument.

Intuition:

$$\begin{aligned}\frac{d}{da}f(x_1^*(a), \dots, x_N^*(a); a) &= \frac{\partial f(x_1^*(a), \dots, x_N^*(a); a)}{\partial x_1^*(a)} \underbrace{\frac{\partial x_1^*(a)}{\partial a}}_{=0} + \dots + \frac{\partial f(x_1^*(a), \dots, x_N^*(a); a)}{\partial a} \\ &= \frac{\partial f(x_1^*(a), \dots, x_N^*(a); \textcolor{red}{a})}{\partial \textcolor{red}{a}}\end{aligned}$$

$x_i^*(a)$ are solutions to the problem \rightarrow increasing a has no effect on each $x_i^*(a)$ \rightarrow only direct partial effect of a matters

7.2 Example: Shephard's Lemma (Prof. Harms: Int'l Trade)

Harms/International Trade

Supply



- The firm's cost minimization problem (contd.):

- Shephard's Lemma:

$$\frac{\partial c_i}{\partial \omega_j} = v_{ij}(\omega, y_i)$$

Note: This is an application of the **Envelope Theorem**.

(The cost function is an **optimal-value function**, i.e. it is based on solution of an optimization problem. As a consequence, only the *direct* effect of a (marginal) change in factor prices on costs has to be considered.)

Harms/International Trade

Supply

- The firm's cost minimization problem (contd.):
 - The solution to the firm's cost minimization problem yields the **cost function**

$$c_i = c_i(\omega, y_i) = \sum_{j=1}^M \omega_j v_{ij}(\omega, y_i) \quad \text{— Conditional demand for factor } j$$

The *minimal* costs of producing y_i given the vector of factor prices ω .

- **Marginal costs:** $MC_i = \frac{\partial c_i}{\partial y_i}$ **Average costs:** $AC_i = \frac{c_i}{y_i}$
- With constant returns to scale

$$c_i(\omega, y_i) = \varphi_i(\omega) \cdot y_i$$

Implication: For given factor prices, $MC_i = AC_i$ independent of output level.

2-28

7.3 Exercises

Using the solution to the cost minimisation of Section 4, i.e. $\frac{w}{r} = \frac{\partial f(K^*, L^*) / \partial L}{\partial f(K^*, L^*) / \partial K}$, and the definition of the cost function $c = K^*r + L^*w$ (where K^* and L^* are the conditional factor demands), use the following functional forms to rewrite the cost function as $c = \phi(w, r)y$, where $\phi(\cdot)$ is some function of factor prices.

1. Using a Cobb-Douglas function, $y = f(K, L) = K^\alpha L^{1-\alpha}$, where $0 < \alpha < 1$.
2. Using a CES function, $y = f(K, L) = [\alpha K^\theta + (1 - \alpha)L^\theta]^\frac{1}{\theta}$, where $0 < \alpha < 1$ and $\theta < 1$.

Show that marginal costs $\partial c / \partial y$ equals average costs c / y for

3. the Cobb-Douglas case.
4. the CES case.

Part V

Intertemporal Maximisation Problem

8 Discrete time model

8.1 Setup

Consider a typical macroeconomic model within this framework, the **Social Planner Problem**:

$$\max_{\{C_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(C_t)$$

for $0 < \beta < 1$, subject to

$$K_{t+1} = f(K_t, C_t)$$

where $f(\cdot)$ is some function of the current capital stock and current consumption.

- This problem can be solved using the Lagrangian method,

$$\mathbb{L}(C_t, K_{t+1}) = \sum_{t=0}^{\infty} \beta^t \{u(C_t) + \lambda_t [f(K_t, C_t) - K_{t+1}]\}$$

Taking the FOCs w.r.t. C_t and K_{t+1} yields

$$\begin{aligned} \frac{\partial \mathbb{L}}{\partial C_t} &= \beta^t \{u'(C_t) + \lambda_t f_{C_t}(K_t, C_t)\} = 0 \\ \Leftrightarrow -\frac{u'(C_t)}{f_{C_t}(K_t, C_t)} &= \lambda_t \\ \frac{\partial \mathbb{L}}{\partial K_{t+1}} &= \beta^t \{\lambda_t [-1]\} + \beta^{t+1} \{\lambda_{t+1} f_{K_{t+1}}(K_{t+1}, C_{t+1})\} = 0 \\ \Leftrightarrow \lambda_t &= \beta \lambda_{t+1} f_{K_{t+1}}(K_{t+1}, C_{t+1}) \end{aligned}$$

Combining both FOCs together yields the Euler equation

$$u'(C_t) = \beta f_{K_{t+1}}(K_{t+1}, C_{t+1}) u'(C_{t+1}) \frac{f_{C_t}(K_t, C_t)}{f_{C_{t+1}}(K_{t+1}, C_{t+1})}$$

8.2 Example: Sovereign default (Prof. Sauré: Fin. Economics I)

Costs of Sovereign default – 2. Exclusion from financial markets – theory

Scenario 1 – no risk of default (very high default costs), the efficient benchmark:

The country can fully commitment to repay and solves

$$\max_{c_t, d_t} \sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to

$$d_t = c_t - y_t + (1 + r)d_{t-1}$$

and

$$\lim_{T \rightarrow \infty} \frac{d_T}{(1 + r)^T} = 0 \quad (\text{No Ponzi condition})$$

8.3 Exercises

1. What is the optimal consumption rule for the Sovereign default model when utility is CRRA, i.e. $u(c_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma}$? With $\sigma \geq 0$ and $\sigma \neq 1$.
2. What is the optimal consumption rule for the Sovereign default model when utility is exponential, i.e. $u(c_t) = \frac{e^{\alpha c_t}}{\alpha}$? With $\alpha > 0$.

9 Continuous time problem

9.1 Setup

Continuous time problem

$$\max_{\{c(\tau)\}_{\tau=t}^{\infty}} U(t) = \int_t^{\infty} e^{-\rho[\tau-t]} u(c(\tau)) d\tau$$

subject to

$$\dot{a}(\tau) = r(\tau) a(\tau) + w(\tau) - c(\tau)$$

- This problem can be solved using the Hamiltonian approach
- This approach is related to the Lagrangian, but is specific to continuous time models.
- The steps are similar: set up your objective function, take first-order conditions, and solve for the optimal rule.

Setting up the Hamiltonian here gives us

$$\mathbb{H} = u(c(\tau)) + \lambda(\tau)[r(\tau)a(\tau) + w(\tau) - c(\tau)].$$

- Thus, we maximise the instantaneous function of the control variable in blue with respect to the equation describing the evolution of the state variable in red.
- $\lambda(\tau)$ is the shadow price as before.
- The FOCs are now given by:

$$\begin{aligned}\frac{\partial \mathbb{H}}{\partial c} &= 0 \Leftrightarrow u'(c(\tau)) = \lambda(\tau) \\ \frac{\partial \mathbb{H}}{\partial a} &= \rho\lambda(\tau) - \dot{\lambda}(\tau) \Leftrightarrow \dot{\lambda}(\tau) = (\rho - r(\tau))\lambda(\tau)\end{aligned}$$

- The first FOC looks at the optimal choice of the control variable such that the Hamiltonian does not change any more.
- The second FOC looks at the optimal change of the state variable w.r.t. the Hamiltonian such that it equates the difference between the flow value of the shadow price ($\rho\lambda$) and the change of the shadow price over time ($\dot{\lambda}$)
- Next, we take the derivative of the first FOC w.r.t. time, and then plug in the result and the first FOC itself into the second FOC, and rearrange.

9.2 Example: the central planner (Prof. Wälde: Advanced Macro)

- Preferences

- We study a central planner problem
- This allows us to focus on optimally chosen s in the simplest way (as opposed to decentralised economy)
- Objective function is social welfare function $U(t)$

$$U(t) = \int_t^{\infty} e^{-\rho[\tau-t]} u(C(\tau)) d\tau \quad (3.5)$$

where

- $u(C(\tau))$ is the instantaneous utility function,
- $\rho > 0$ is the time preference rate (measuring impatience of individual) and
- $e^{-\rho[\tau-t]}$ is the discount factor (or function)
- Planning/ optimal behaviour starts in t (like 'today') and goes up to infinity ∞

- Instantaneous utility

- Functional form ...

$$u(C(\tau)) = \frac{(C(\tau))^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0 \quad (3.6)$$

- ... implies constant intertemporal elasticity of substitution of $1/\sigma$

- (Note that the logarithmic utility function is a special case of this functional form,

$$\lim_{\sigma \rightarrow 1} u(C(\tau)) = \ln C(\tau),$$

apply L'Hôpital's rule to find this)

- The resource constraint

- Maximization problem becomes meaningful only with a constraint

- Constraint here is a resource constraint as we look at the economy as a whole

- Capital evolves according to

$$\dot{K}(t) = Y(K(t), L) - \delta K(t) - C(t) \quad (3.7)$$

- The change in the capital stock (net investment) is given by output minus consumption (gross investment) minus depreciation $\delta K(t)$

3.14

9.3 Exercises

1. What is the optimal consumption rule (or Keynes-Ramsey rule), when production is Cobb Douglas, $Y(K(t), L) = K(t)^\alpha L^{1-\alpha}$? With $0 < \alpha < 1$.
2. What is the optimal consumption rule (or Keynes-Ramsey rule), when production is CES, $Y(K(t), L) = [\alpha K(t)^\theta + (1 - \alpha)L^\theta]^\frac{1}{\theta}$? With $\theta < 1$.

Part VI

Ordinary Least Squares

10 Typical OLS regression

10.1 Preliminaries: method of moments for 1 explanatory variable

Typical OLS regression equation explains a (linear) relationship between a **dependent** variable y and an **independent** variable x (see “Intro Econometrics” by Prof. Dr. Van Ewijk + “Cross Section and Panel Data” by Prof. Dr. Schank for multivariate case)

$$y = \beta_0 + \beta_1 x + u,$$

where β_0 is the intercept, β_1 is the coefficient of interest in the regression, and u is the residual.

In turn, we estimate β_0 and β_1 by solving two equations, given a sample size n ,

$$\begin{aligned}\frac{1}{n} \sum_{i=1}^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) &= 0, \\ \frac{1}{n} \sum_{i=1}^n x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i) &= 0.\end{aligned}$$

We can rewrite the first expression as

$$\bar{y} = \hat{\beta}_0 + \hat{\beta}_1 \bar{x} \Leftrightarrow \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x},$$

where \bar{y} and \bar{x} are the sample averages of y and x . Thus, if we know $\hat{\beta}_1$, we can compute the estimate of the intercept (the sample averages being trivial to obtain).

To obtain this estimate of the slope, we go back to the second expression above, and replace $\hat{\beta}_0$ with our expression in terms of sample averages,

$$\sum_{i=1}^n x_i (y_i - [\bar{y} - \hat{\beta}_1 \bar{x}] - \hat{\beta}_1 x_i) = 0,$$

where we drop $\frac{1}{n}$, since the other side is equal to 0, and thus it does not affect the solution. We can then rewrite this in terms of deviations from the sample averages (i.e. deviations from the

mean),

$$\sum_{i=1}^n x_i[y_i - \bar{y}] = \hat{\beta}_1 \sum_{i=1}^n x_i[x_i - \bar{x}].$$

Using properties of sums we can rewrite this as (see Wooldrige (2000) for details)

$$\sum_{i=1}^n [x_i - \bar{x}][y_i - \bar{y}] = \hat{\beta}_1 \sum_{i=1}^n [x_i - \bar{x}]^2 \Leftrightarrow \hat{\beta}_1 = \frac{\sum_{i=1}^n [x_i - \bar{x}][y_i - \bar{y}]}{\sum_{i=1}^n [x_i - \bar{x}]^2}.$$

10.2 Example: Multiple Regression (Prof. van Ewijk: Intro Econometrics)

Recall: multiple regression

General format:

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \dots + \beta_k X_{ki} + u_i, \quad i = 1, \dots, n$$

e.g. We estimate: $Health_i = \beta_0 + \beta_1 Exercise_i + \beta_2 Education_i + u_i$, in which:

- $Health$ = Some variable measuring health
 - $Exercise$ = Amount of exercise a person does
 - Variant A: measured in hours per week
 - Variant B: dummy variable indicating whether person does exercise yes/no
 - $Education$ = Number of years of education
 - Result:
$$\widehat{Health}_i = 5.4 + 2.00 * Exercise_i + 0.05 * Education_i$$

(0.9)

(0.50)

(0.20)
1. Interpret each of the 6 numbers – for variant A & B
 2. Which parameters are significant?
 3. I was interested *only* in the effect of exercise on health. Why would I have chosen to include the variable $Education$?

10.3 Exercises

Given the following samples, compute the estimates of the intercept $\hat{\beta}_0$ and of the slope $\hat{\beta}_1$.

1. The following table gives the number of units produced for a clone army y (in thousands) and the number of hours spent per day producing clones x over a 10-day period on Kamino.

y	5	8	15	12	5	8	2	10	12	17
x	1.5	2	2.5	2.5	2	2	1	1.5	2	2.5

Given your estimates, how many clones would be produced if the Kaminoans spent 4h/day producing them?

2. The following are final grades in an exam y and average number of hours spent studying per week x for 12 students.

y	51	45	44	43	40	35	33	24	23	21	16	2
x	10	12	10	6	9	9	9	7	5	6	4	2

Given your estimates, what grade would a student achieve if they spent 11h/week studying toward their exam?

Part VII

Probability Theory

11 Distribution

11.1 Discrete random variable

We begin by defining the probability that a random variable X take on a specific value x as

$$f_X(x_i) := Prob(X = x_i) = p_i,$$

where we call $f_X(x_i)$ the probability mass function (pmf) of X . Correspondingly, we can define the cumulative probability that X take on any value less than or equal to x_i as

$$F_X(x_i) := Prob(X \leq x_i) = \sum_{x_i \leq x} f_X(x_i),$$

also called the cumulative distribution function (cdf).

11.2 Continuous random variable

Consider a continuous random variable X that can take any value in the interval $S = [a, b]$. The probability that X be smaller than or equal to some $x \in S$ is given by

$$F_X(x) := \text{Prob}(X \leq x),$$

which is its CDF. Correspondingly, we can compute the probability density function (pdf) of X as

$$f_X(x) = \frac{\partial F_X(x)}{\partial x}.$$

Thus, the probability that X takes on a value between points c and d such that $a < c < d < b$ is given by

$$\text{Prob}(c < X < d) = F_X(d) - F_X(c) = \int_c^d f_X(x) dx.$$

Notation is often abbreviated to $f(x) := f_X(x)$, when there is no ambiguity about the domain of the variable.

11.3 Example: Distribution of discrete and continuous RV (Prof. Schank: Stats Methods)

Introduction

Probability

Inductive Statistics

Monte Carlo Simulation

Distribution of discrete random variables

- ▶ **Discrete random variable** X : takes only a finite number of different values $\{x_1, x_2, \dots, x_i, \dots, x_k\}$.
- ▶ Each of these values is realized with a certain probability $\{p_1, p_2, \dots, p_i, \dots, p_k\}$.
- ▶ The **probability mass function (PMF)** of the RV is defined for all real numbers x :

$$\begin{aligned} f_X(x) &= \Pr(X = x) \\ &= \begin{cases} p_i & \text{if } x = x_i \\ 0 & \text{if } x \notin \{x_1, \dots, x_k\} \end{cases} \end{aligned}$$

20 / 60

Distribution of discrete RV

- The **cumulative distribution function (CDF)** is

$$\begin{aligned} F_X(x) &= \Pr(X \leq x) \\ &= \sum_{x_i \leq x} p_X(x_i) \end{aligned}$$

Distribution of continuous RV

- ▶ **Continuous random variable X** : takes an infinite number of different values $x \in \mathbb{R}$ (e.g. body height)
- ▶ Each of these values x is realized with probability $\Pr(X = x) = 0$.
- ▶ The **cumulative distribution function (CDF)** is:

$$F_X(x) = \Pr(X \leq x)$$

- ▶ The **probability density function (PDF)** is:

$$f_X(x) = \frac{\partial F_X(x)}{\partial x}$$

- ▶ The probability for a number between a and b is

$$\Pr(a \leq X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x) dx$$

11.4 Exercises

1. Using a fair 20-sided die, what is the probability that you will roll a 17? What is the probability that you will roll a 15 or higher?
2. Given a uniformly distributed variable X on the interval $[a, b]$ with CDF, $F(x) = \frac{x-a}{b-a}$, compute the PDF of this function. Additionally, what is the probability that $x = c$, where $c = \frac{b-a}{3}$? What is the probability that X is greater than c ?

12 Mean and variance

12.1 Discrete random variable

Consider a discrete random variable X that can take any value x_1, x_2, \dots, x_N with corresponding probabilities p_1, p_2, \dots, p_N . The expected value (or mean) of random variable X is

$$\mathbb{E}(X) \equiv \mu_X = \sum_{i=1}^N p_i x_i$$

The variance is defined as

$$\text{Var}(X) \equiv \sigma_X^2 = \sum_{i=1}^N p_i (x_i - \mu_X)^2$$

The standard deviation is the square root of the variance

$$\sigma_X = \sqrt{\sum_{i=1}^N p_i (x_i - \mu_X)^2}$$

12.2 Continuous random variable

Given Section 11, we can write,

- the expected value of X is

$$\mathbb{E}(X) = \int_a^b x f(x) dx,$$

- and the variance of X is

$$\text{Var}(X) = \int_a^b (x - \mathbb{E}(X))^2 f(x) dx.$$

Of course, the standard deviation is also given by the square root of the variance,

$$\sigma_X = \sqrt{\text{Var}(X)}.$$

12.3 Example: Mean and Variance of discrete and continuous RV (Prof. Schank: Stats Methods)

Important features of distributions

- **Expected value** (mean): measure of central tendency

$$\begin{aligned}\mu_X = E(X) &= \sum_{j=1}^k x_j \cdot f_X(x_j) \quad \text{if discrete} \\ &= \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx \quad \text{if continuous}\end{aligned}$$

- **Variance**: measure of variability

$$\begin{aligned}\sigma_X^2 = \text{Var}(X) &= E((X - \mu_X)^2) \\ &= E(X^2) - \mu_X^2\end{aligned}$$

12.4 Exercises

1. Let $X \sim \log(k)$ (i.e. logarithmically distributed), compute the expected value of the discrete r.v. X (Note: the pmf of the log-distribution is $p_i(x_i) = \frac{-1}{\ln(1-k)} \frac{k^{x_i}}{x_i}$, where $x_i \in \{1, 2, 3, \dots\}$ is the support of the distribution and $0 < k < 1$ is a parameter).
2. Let $Y \sim \mathcal{N}(0, 1)$ (i.e. standard normally distributed), compute the expected value of the continuous r.v. y . (Note: the pdf of the standard normal distribution is given by: $f(y) = \frac{1}{\sqrt{2\pi}} e^{-(y^2)/2}$, and Y is defined on the entire real line).
3. Let $Z \sim \text{uniform}(0, 2)$ (i.e. uniformly distributed), compute the expected value AND the variance of the continuous r.v. Z . (Note: the pdf of this uniform distribution is given by: $f(z) = \frac{1}{2}$, and Z is defined on the set of real numbers from 0 to 2).