Capital Income Risk and the Dynamics of the Wealth Distribution

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1. Introduction

- In recent years, there has been a rising concern about an increase in inequality, i.e. a concern about changes in wealth distributions over time
- Indirect evidence from the success of Piketty’s (2014) “Capital in the Twenty-First Century”

Understanding the determinants of the wealth distribution and its evolution over time is of enormous academic and public interest
1. Introduction

Our questions

- How can we explain the *evolution* of wealth distributions?
- Should the density of wealth or wealth shares be targeted?

Our framework

- We work with a partial equilibrium model
- We allow for
  - idiosyncratic labour income risk,
  - *idiosyncratic capital income risk and*
  - *ex-ante heterogeneity in financial ability*

![The evolution of the wealth distribution of the NLSY79 cohort](image)

**Figure 1** *The evolution of the wealth distribution of the NLSY79 cohort*
1. Introduction

Existing literature related to our questions

- Determinants of a *cross-sectional* distribution of wealth (with a fat right tail)
  - idiosyncratic labour income risk with “superstar state”: Castañeda et al. (2003)

- The dynamics of distributions
  - Bayer and Wälde (2010a) study the dynamics of wealth distributions in a Bewley-Huggett-Aiyagari model (without capital income risk)
  - Gabaix et al. (2016) study the dynamics of *income* inequality
  - Kaymak and Poschke (2016) present how top 1%, 5% and 10% wealth shares evolve over time (without capital income risk)
  - Benhabib, Bisin and Luo (2017, sect. 6) study how wealth shares evolve over time (with capital income risk)
1. Introduction

Preview of findings

Figure 2  *The evolution of the wealth density in the data and in the model when targeting the 2008 wealth density*

- When targeting the wealth density, the fit is 96.1% and the fit for (non-targeted) Lorenz curve is 92%.
- When targeting the Lorenz curve, the fit is 99.5% and the fit for the (non-targeted) wealth density is 74.5%.
1. Introduction

Structure of the talk

2. The model
3. Fokker-Planck equations
4. Matching the evolution of wealth densities
5. Robustness checks
6. Concluding remarks
2. The model

- We consider an individual, who
  - faces idiosyncratic labour income risk
    \[ dz(t) = gz(t) \, dt + \left[ w(\Gamma(t)) - z(t) \right] dq_\mu(t) + \left[ b(\Gamma(t)) - z(t) \right] dq_s(t) \]  
    where \( \Gamma(t) \equiv \Gamma_0 e^{gt} \),
  - faces idiosyncratic interest rate risk
    \[ dr(t) = \left[ r_{\text{high}} - r(t) \right] dq_{\text{low}}(t) + \left[ r_{\text{low}} - r(t) \right] dq_{\text{high}}(t) \]  
    where \( q_\mu, q_s, q_{\text{low}}, \) and \( q_{\text{high}} \) are four Poisson processes with positive arrival rates \( \mu, s, \lambda_{\text{low}}, \) and \( \lambda_{\text{high}}, \) respectively,
  - is financially constrained according to
    \[ da(t) = \left\{ r(t) \, a(t) + z(t) - c(t) \right\} \, dt. \]
  - has preferences represented by a CRRA utility function
    \[ u(c(t)) = \begin{cases} 
    \frac{c(t)^{1-\sigma} - 1}{1-\sigma}, & \sigma > 0, \sigma \neq 1, \\
    \ln(c(t)), & \sigma = 1.
    \end{cases} \]
2. The model

- We assume
  - labour income grows at a rate of $g$ (in order to match wealth levels)
  - changes in the interest rate are not anticipated
  - each individual draws the (arrival) rates of the jumping processes from a distribution before becoming economically active ⇒ the ex-ante heterogeneity in financial ability (including type dependence and scale dependence as Gabaix et al. (2016))
  - there is minimum consumption level
    \[ c_{\min}(t) = \zeta b(t), \quad \zeta \in (0, 1) \] (5)

- there is a natural borrowing limit
  \[ a_{\text{nat}}(t) = -\frac{(1 - \zeta) b(t)}{r_{\text{high}} - g} \] (6)
2. The model

- Optimal behaviour is described by two Keynes-Ramsey rules, which describe how optimal consumption evolves over time when employed and when unemployed

\[
dc_r^w (a) = \frac{c_r^w (a)}{\sigma} \left\{ r - \rho + s \left[ \left( \frac{c_r^w (a)}{c_r^b (a)} \right)^\sigma - 1 \right] \right\} dt \\
+ \left[ c_r^b (a) - c_r^w (a) \right] dq_s. \tag{7a}
\]

\[
dc_r^b (a) = \frac{c_r^b (a)}{\sigma} \left\{ r - \rho - \mu \left[ 1 - \left( \frac{c_r^b (a)}{c_r^w (a)} \right)^\sigma \right] \right\} dt \\
+ \left[ c_r^w (a) - c_r^b (a) \right] dq_\mu. \tag{7b}
\]
2. The model

- When we detrend the KRRs, we obtain a two-dimensional ODE system for optimal consumption

\[
\frac{d\hat{c}_r^\hat{w}}{d\hat{a}} = \frac{\frac{r-\rho}{\sigma} - g + \frac{s}{\sigma} \left[ \left( \frac{\hat{c}_r^\hat{w}(\hat{a})}{\hat{c}_r^\hat{b}(\hat{a})} \right)^\sigma - 1 \right]}{(r - g) \hat{a} + \hat{w} - \hat{c}_r^\hat{w}(\hat{a})} \hat{c}_r^\hat{w}(\hat{a}) ,
\]

(8a)

\[
\frac{d\hat{c}_r^\hat{b}}{d\hat{a}} = \frac{\frac{r-\rho}{\sigma} - g - \frac{\mu}{\sigma} \left[ 1 - \left( \frac{\hat{c}_r^\hat{b}(\hat{a})}{\hat{c}_r^\hat{w}(\hat{a})} \right)^\sigma \right]}{(r - g) \hat{a} + \hat{b} - \hat{c}_r^\hat{b}(\hat{a})} \hat{c}_r^\hat{b}(\hat{a}) .
\]

(8b)
2. The model

- Phase diagrams

Low interest rate regime 
\[ r < \rho + \sigma g \]

High interest rate regime 
\[ r > \rho + \sigma g \]
3. Fokker-Planck equations

- We are interested in the joint density of wealth and time \( g(a, t) \)
- We derive a system of two partial differential equations (because of two-state employment status) describing how wealth densities change over time (when employed and when unemployed)
  - If time was fixed, we would obtain the cross-sectional distribution of wealth
  - If the level of wealth was fixed, then the equations would tell us how the share of population owning such a fixed level of wealth or less evolves over time
- These are called *Fokker-Planck equations* (or Kolmogorov forward equations)
- We solve the FPEs by the methods of characteristic (Nagel 2013)
3. Fokker-Planck equations

\[
\frac{\partial}{\partial t} p^\hat{w} (\hat{a}, t) + \left[ (r - g) \hat{a} + \hat{w} - \hat{c}_r \hat{w} (\hat{a}) \right] \frac{\partial}{\partial \hat{a}} p^\hat{w} (\hat{a}, t)
= \left[ \frac{d\hat{c}_r \hat{w} (\hat{a})}{d\hat{a}} - (r - g) - s \right] p^\hat{w} (\hat{a}, t) + \mu p^\hat{b} (\hat{a}, t), \tag{9a}
\]

\[
\frac{\partial}{\partial t} p^\hat{b} (\hat{a}, t) + \left[ (r - g) \hat{a} + \hat{b} - \hat{c}_r \hat{b} (\hat{a}) \right] \frac{\partial}{\partial \hat{a}} p^\hat{b} (\hat{a}, t)
= sp^\hat{w} (\hat{a}, t) + \left[ \frac{d\hat{c}_r \hat{b} (\hat{a})}{d\hat{a}} - (r - g) - \mu \right] p^\hat{b} (\hat{a}, t), \tag{9b}
\]

- Remarks
  - \( p(\hat{a}, t) = p(\hat{a}, t | \hat{z} = \hat{w}) p(\hat{z} = \hat{w}) + p(\hat{a}, t | \hat{z} = \hat{b}) p(\hat{z} = \hat{b}) = p^\hat{w} (\hat{a}, t) + p^\hat{b} (\hat{a}, t) \)
  - \( \hat{c}_r^i (\hat{a}), i \in \{ \hat{b}, \hat{w} \}, \) is the optimal consumption resulted from the ODE system (8)
  - The density of wealth with trend is \( g(a, t) = \frac{p(a(t)/\Gamma(t), t)}{\Gamma(t)} \)
4. Matching the evolution of wealth densities

- Matching the evolution of wealth densities means
  - starting from an *initial density* and
  - matching a targeted density at a future point in time or
  - matching multiple densities simultaneously

- We use the data on wealth extracted from NLSY79 that contains 12 cross sections of wealth

- Parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>$\mu$</td>
<td>21.99%</td>
</tr>
<tr>
<td>$s$</td>
<td>1.19%</td>
</tr>
<tr>
<td>$\hat{w}$</td>
<td>2280.8$</td>
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<tr>
<td>$g$</td>
<td>3.4%</td>
</tr>
<tr>
<td>$\hat{b}/\hat{w}$</td>
<td>30%</td>
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<tr>
<td>$\zeta$</td>
<td>97%</td>
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<tr>
<td>$\rho$</td>
<td>1%</td>
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<tr>
<td>$\sigma$</td>
<td>1</td>
</tr>
<tr>
<td>$r_{\text{low}}$</td>
<td>3.5%</td>
</tr>
<tr>
<td>$r_{\text{high}}$</td>
<td>4.5%</td>
</tr>
</tbody>
</table>
4. Matching the evolution of wealth densities

- We target the 2008 empirical density of wealth. How?
  - Use the 1986 empirical density as an initial condition
  - Solve the FPEs (9) for each financial type to obtain two densities $g_j(a, 2008)$ and $g_{j+1}(a, 2008)$ (each financial type might start with low or high interest rate)
  - Attach a probability $p_j$ to each density $g_j(a, 2008)$
  - Choose probabilities $p_j$ to minimize the area between the model density and the empirical density, i.e. to maximize the following measure of fit

$$F(2008) = 1 - \frac{\int_{-\infty}^{\infty} \left| g_{\text{model}}(a, 2008) - g_{\text{data}}(a, 2008) \right| \, da}{2}. \quad (10)$$

where

$$g_{\text{model}}(a, 2008) = \sum_{j=1}^{2n} p_j g_j(a, 2008),$$

and the integral is the $L^1$-norm or the total variation norm.

- Iterate these steps for $n = 2, \ldots, 131$
4. Matching the evolution of wealth densities

- The optimal number of financial types is 30 (60 interest rate paths)
- Wealth density in 2008 is (almost) perfectly matched by model

![Wealth Density Graph](image)

**Figure 3** The wealth distribution in 2008 in the data and in the model

- The (implied) mean return is 4.3% and the standard deviation is 0.42%
- While the mean return is empirically plausible, the standard deviation seems to be too low
  ⇒ this baseline calibration seems to “overexplain” the dynamics of the wealth distribution
4. Matching the evolution of wealth densities

**Figure 4** The evolution of the wealth density in the data and in the model when targeting the 2008 wealth density

- Quantitative measure of fit

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<tbody>
<tr>
<td>Value</td>
<td>100</td>
<td>73.5</td>
<td>63.0</td>
<td>60.6</td>
<td>61.4</td>
<td>66.3</td>
<td>72.1</td>
<td>77.2</td>
<td>81.9</td>
<td>84.4</td>
<td>87.4</td>
<td>96.1</td>
</tr>
</tbody>
</table>

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5. Robustness checks

- The goodness of fit is attributed to both labour income risk and capital income risk.
- We therefore inquire into the role of pure labour income risk, pure capital income risk and the financial types.

We find

- The fit,
  (i) when there is only labour income risk, is below 30%.
  (ii) when there is only capital income risk, is below 66%.
  (iii) when there is only capital income risk and when we allow for “awesome” or “superstar” realizations of the interest rate, increases to 96.7%.
  (iv) when there is only one financial type (average), is below 60%.
5. Robustness checks

(i) (ii)

(iii) (iv)

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September 13, 2019  19 / 28
5. Robustness checks

- When targeting the 2008 density (96.1%), the non-targeted Lorenz curve in the data is matched (92%) by model quite well (top figures).
- But when targeting the Lorenz curve (99.5%), the density fit (74.5%) is not very convincing (bottom figures).
- This shows a strong trade-off between fitting densities and Lorenz curves.
6. Concluding remarks

Our questions

- How can we explain the evolution of the wealth distribution?
- Should the density of wealth or wealth shares be targeted?
6. Concluding remarks

Our answers

- For an empirically plausible distribution of interest rate, both
  - labour income risk and
  - capital income risk
  are essential for explaining the dynamics of the wealth distribution

- With interest rate distributions including “awesome” or “superstar states”, capital income risk alone can generate extremely good fits for the dynamics of the wealth distribution

- Targeting density seems to produce a better overall fit if one would like to obtain a good fit of both the density and the wealth shares
Thank you!