Capital Income Risk and the Dynamics of the Wealth Distribution

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Labour income follows a deterministic growth trend and fluctuates between two values. Interest rates are drawn initially, fluctuate between two values and can differ in their arrival rates. Low interest rates imply a stationary long-run wealth distribution, high interest rates imply exploding wealth dynamics. When matching the NLSY 79 evolution of the wealth distribution from 1986 to 2008, we obtain a fit of 96.1%. With a more flexible interest rate distribution, employing "superstar states", the fit can increase to 96.7%. For the fit of 96.1%, the standard deviation of model returns is much lower than the empirical standard deviation.

JEL Codes: C6, D3, E2

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1 Introduction

[Motivation] Understanding wealth distributions has always been of major academic and public interest. Concerns about efficiency and (in-) equality are central to this interest. In recent years, there has been a rising concern about an increase in inequality; that is, a concern about changes in wealth distributions over time. This suggests that understanding the determinants of wealth distribution and especially its evolution over time is of enormous academic and public interest.

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[The open issue] While there is quite some research on the distribution of wealth (see below), very little is known about how quickly it changes over time. Therefore, this paper quantitatively asks: What are the necessary building blocks of an explanation of the dynamics of wealth? To make this question precise, we ask: Under which conditions can a relatively standard model of idiosyncratic risk with standard parameter values match the evolution of the wealth distribution of the National Longitudinal Survey of Youth 1979 (NLSY 79) from 1986 to 2008?

[The setup] Individuals face uncertain labour income as they stochastically move back and forth between employment and unemployment. Individuals can self-insure against implied consumption fluctuations by accumulating wealth. When unemployed, the individual's maximum debt level is given by a natural borrowing constraint. Individuals also face uncertain returns on their wealth. The return fluctuates randomly between two values.[2] The transition rates between these values can differ between individuals, which we describe as an individual's 'financial type'. Each individual draws their type before entering the labour market. Individuals with a high financial ability will experience high returns more frequently than individuals with a lower financial ability. Because we want to understand one cohort of the US population, we work with a partial equilibrium model.[3] The wage, unemployment benefits, the growth rate of the wage and benefits and the distribution of returns are exogenous.

The quantitative analysis is facilitated by the use of Fokker-Planck equations (FPEs). Treating wealth as a continuous variable, our FPEs take the form of a partial differential equation system that describes the evolution of the wealth distribution over time. They can be derived from the fundamentals of the model, taking optimal consumption behaviour of agents into account.

In our calibration, we compute the average wage, wage growth, the arrival rates of jobs and the separation rate from the NLSY data. In our baseline calibration, we set the time preference rate to 1% and the degree of risk aversion to 1. Our idiosyncratic interest rate fluctuates between annual values of 3.5% and 4.5%. We perform various robustness analyses.

[Findings] Our main contribution lies in demonstrating the quantitative usefulness of FPEs for understanding the *evolution* of *entire* distributions. We apply this tool to understand the relative importance of capital income risk (as in Benhabib, Bisin and Zhu, 2011) vs. labour income risk to match the evolution of NLSY 79 wealth densities from 1986 to 2008. We show that for interest rate distributions with "awesome" or "superstar" states,[4] we can match the

[2] To simplify the numerical analysis, we assume individuals are myopic with respect to *changes* in the interest rate. Optimal policy functions are a function (inter alia) of the *level* of the interest rate. See footnote 29 for further discussion.

[3] Our model below could also be seen as describing a small open economy with free international capital flows (see Bayer et al. 2019). As we focus on one this one NLSY cohort, the partial equilibrium interpretation seems more appropriate.

[4] Labour income states with low probability but very high labour income where employed by Castaneda et

wealth density almost perfectly. For an empirically convincing interest rate and labour income distribution, we find that both capital income risk (including type and scale dependence as in Gabaix et al., 2016) and labour income risk are needed to obtain a fit of above 96%.

In more detail, our findings are as follows: The empirical distribution of wealth in 2008 has more probability mass to the right and to the left than the empirical distribution in 1986 when individuals entered the labour market at the age of 21 to 28. We start by targeting the 2008 wealth distribution, using the 1986 distribution as the initial condition. According to our measure of fit (focusing on densities directly rather than on wealth shares), the model density in 2008 covers 96.1% of the empirical density.

To obtain this result, the low realization of the return (at 3.5%) needs to lie below the threshold level, such that a stationary distribution obtains.[5] Together with a parameter determining the minimum consumption level at the natural borrowing limit, this allows us to assure that the left tail of the wealth distribution converges in 22 years from the initial distribution to the density in 2008. The high realization (set at 4.5%) needs to lie above this threshold level, yielding exploding wealth dynamics. Employing this "exploding regime" (Benhabib and Bisin, 2018) is essential to obtain the fat right tail (i.e. high Pareto coefficient) in the distribution of wealth in 2008.

This fit of 96% for our baseline model is obtained with 30 different financial types. The share of individuals in the NLSY cohort belonging to each of these financial types is computed and lies between 0.3% to 8.0%. Technically, we compute the weights of a mixture of densities from different types in the model by minimizing the absolute distance between our model density and the empirical density. We can obtain similarly good measures of fits for other target years than 2008. When we target all years jointly (i.e. when the evolution of the wealth distribution is taken into account for each year with empirical wealth information), the average fit over all years is higher (at 88.9%) than the average fit when targeting one specific year.

Our robustness checks first analyse the relative importance of capital income risk to labour income risk. We show that one can obtain similarly good findings as in the baseline model when abstracting from any labour income risk. In various "pure capital risk" specifications, we obtain fits of up to 96.7%. Returning to the baseline model, an increase of the high interest rate (from 4.5% to 8%) hardly increases the fit for 2008 (to 97.3%). A lower degree of risk aversion (0.8)

al. (2003) to obtain sufficient wealth inequality (Hubmer et al, 2017, Benhabib et al., 2018). These states are sometimes referred to as "superstar" or "awesome" states. Here, we transfer these terms to describe an idiosyncratic interest rate distribution which has a mean that exceeds the mean of empirical interest rate distributions.

[5] In models with idiosyncratic income risk without economic growth, there is a stationary distribution if the interest rate lies below the time preference rate, $r < \rho$. Because we allow for growth, there is a stationary distribution in our setup if the interest rate lies below the time preference rate plus the product of risk aversion and the growth rate, $r < \rho + \sigma g$.

worsens the overall fit (to 90.3%), a higher degree (1.2) implies that the fit falls dramatically (to 44.7%) as the exploding regime vanishes. Given that we target densities directly rather than wealth shares, we also inquired into the fit in terms of wealth shares. When we target the average of densities over all years, (non-targeted) wealth shares differ from empirical wealth share on average by 2.6%. When we target wealth shares, the model Lorenz curve coincides with the empirical Lorenz curve of 2008 by 99.5%.

We "test" our baseline model and the pure-capital-risk specifications by comparing the implied standard deviations of the idiosyncratic interest rate in our calibrations with empirical standard deviations reported in the literature. While pure-capital-risk models yield a very high fit, they include what could also be termed "awesome" or "superstar states". The mean return in these pure-capital-risk models is too high compared to empirical means of idiosyncratic risk. When we look at the baseline model, the calibrated mean return is in line with empirical evidence. Yet, it seems that empirical standard deviations are one or two orders of magnitude *larger* than those needed in our model to match the dynamics of the wealth distribution. Interest rate uncertainty joint with labour income uncertainty is, therefore, almost "too successful" in explaining wealth inequality.

We also provide empirical Pareto coefficients for measuring the fatness of the right tail for all waves and compare them to our model. Taking right-truncation into account, the empirical coefficients are similar to those reported in the literature. The (non-targeted) model coefficients are sufficiently close to the empirical ones.

[Related literature] We see our paper (i) in the tradition of the literature that studies one cross-section of wealth, especially with a focus on capital income risk, (ii) sharing features of analyses that look at the dynamics of distributions and (iii) as most closely related to two quantitative studies that investigate into the empirical relevance of capital income risk for the evolution of wealth.

The determinants of one distribution of wealth have been studied for a long time. The conventional view starts from idiosyncratic labour income risk in the Bewley-Huggett-Aiyagari tradition. In the tradition of Castañeda et al. (2003), many authors have successfully replicated empirical wealth distributions. To obtain sufficiently thick right tails in the wealth distribution, usually some (low-probability and very high) labour income state is introduced where the income level is an order of magnitude larger than is empirically plausible. This is what Benhabib and Bisin (2018) and Benhabib, Bisin and Luo (2017) refer to as the "awesome state". It, therefore, seems reasonable to search for other determinants of wealth distributions (Benhabib et al. 2017). As shown in the seminal contribution by Benhabib, Bisin and Zhu (2011), risky idiosyncratic returns are one such highly promising source.[6] In an overlapping generations

[6] There is an earlier work that studies the effects of capital income risk as well. Angeletos and Calvet (2005, 2006) employ CARA preferences and allow for additive endowment risk. Angeletos (2007) does not have additive

framework, the authors analytically show that their stationary distribution has a Pareto distribution in the right tail and that the thickness of the right tail increases in capital income risk.[7] We see our paper in this tradition.[8] We find that capital income risk is a quantitatively necessary ingredient to match the density of wealth over its entire range when an empirically convincing labour income process (i.e. without a 'superstar' or 'awesome' state) is employed.

So far, the dynamics of distributions have received far less study. Gabaix et al. (2016) study the dynamics of *income* inequality.[9] Conventional models do not generate sufficiently high transition speeds to match the empirical rise in top income inequality. They introduce scale and type dependence for income dynamics to obtain transition speeds for distributions that are sufficiently large. We also allow for exogenous types (our financial abilities), which in our framework leads to *endogenous* scale dependence (via the exploding regime). We apply our method to quantify the importance of type and scale dependence. Kaymak and Poschke (2016) present how top 1%, 5% and 10% wealth shares evolve over time. We extend their work inter alia by looking at the entire density and, thereby, at all wealth shares and by studying the effect of capital income risk.[10]

To the best of our knowledge, our paper is the first to employ FPEs to understand the quantitative relevance of capital income risk on the evolution of *entire* wealth densities over time. There are, however, two recent studies that share some of our features, although our paper differs from these studies in many important respects. Benhabib, Bisin and Luo (2019) quantitatively emphasize the importance of capital income risk in addition to persistent earnings inequality and bequests to explain wealth distributions and social mobility patterns. While their main analysis focuses on stationary distributions, their robustness check studies how wealth distributions evolve over time. The length of one period of analysis in their approach is 36 years. They study the dynamics of wealth in the SCF over a length of 45 years. This length

endowment risk but employs Epstein-Zin preferences with CEIS and CRRA. In addition to idiosyncratic capital income risk (housholds own private business), there is a second riskless asset. This leads to closed-form solutions for policy functions (as in Merton, 1971). Krusell and Smith (1998) have stochastic discount factors which play a similar role as stochastic interest rates.

[7] Benhabib, Bisin, and Zhu (2015) extend their findings from Benhabib, Bisin, and Zhu (2011) by looking at an infinite horizon setup. Their infinite horizon setup works with non-negative wealth levels, assumes i.i.d. processes for the interest rates and labour income and analytically studies properties of the stationary distribution.

[8] De Nardi and Fella (2017) provide a comprehensive overview of various determinants of wealth distribution.

[9] In their online Appendix E, they also look at wealth inequality. The underlying maximization problem assumes deterministic labour income, iid capital income and that individuals consume an exogenous fraction of their wealth. We will see later, inter alia, that stochastic labour income is crucial for obtaining empirically convincing descriptions of wealth dynamics.

[10] Bayer et al. (2019) provide proof of the existence and stability of stationary wealth distributions in continuous-time models with labour income risk. Parra-Alvarez et al. (2017) structurally estimate a heterogenous agent model. They focus on the identifiability of parameters and apply their method to the 2013 distribution of wealth in the SCF.

of time is approximated by two time periods in their model. We track each of the 12 wealth distributions in the NLSY data from 1986 to 2008 by our method. Employing a continuous time framework, we can choose the model length such that it coincides with data length. Our analysis also allows for explicit stochastic labour income over time.[11] Our robustness check shows that the absence of one source of uncertainty strongly reduces the model's explanatory power.

Capital income risk is also taken into account by Hubmer et al. (2019). They examine drivers of the rise in wealth inequality in the United States over the last 30 years and find, inter alia, that an increase in tax progressivity would reduce wealth inequality. Methodologically, they study a "perfect-foresight transition experiment" and a "myopic transition experiment". Our numerical procedure does not require us to assume perfect foresight or myopic behaviour with respect to all random events.[12] We also allow for both of our labour income and our interest rate processes to have persistent components. Idiosyncratic interest rate shocks in Hubmer et al. (2019) are of a transitory nature and affect individual behaviour only through their effect on end-of-period wealth ("cash on hand").[13] In our setup, the persistent nature of interest rate shocks, which are a salient empirical property stressed by Fagereng et al. (2018), makes interest rates a state variable of the household's decision problem. The nature of wealth accumulation (the standard regime and the 'exploding' regime) depends on the interest rate level and it is the central mechanism to explain fat right tails. They result from type and scale-dependence (Gabaix et al., 2016) of capital income risk.

Our quantitative approach also offers a new feature, not only in comparison to Hubmer et al. (2019). Most of the papers that we are aware of describe wealth inequality by wealth shares. How large is the share of wealth that the richest x% of the population own? In this tradition, leading to Lorenz curves, growth processes and *absolute* wealth levels do not matter when comparing model predictions with data. One can therefore work with stationary models. Because FPEs describe the evolution of densities of *levels* of random variables, it is natural to look at the densities predicted by the model directly and compare them to empirical densities. To this end, we also allow for a growth process in labour income to make sure that model densities grow in levels sufficiently fast as compared to (real) wealth densities.

[11] Benhabib, Bisin and Luo (2019) emphasize that "r and w are stochastic over generations only: agents face no uncertainty within their life span".

[12] We acknowledge that our partial equilibrium approach helps in this respect as the feedback from mean wealth on aggregate variables is absent. See Pröhl (2017) for a novel numerical procedure that does allow us to compute fully rational equilibria with aggregate shocks and distributions as elements of the state space.

[13] This modelling choice reduces the number of state variables of the individual's maximization problem (cash-on-hand, the persistent component of the earning process and the stochastic discount factor in the spirit of Krusell and Smith, 1998) and makes computation faster. We are grateful to Joachim Hubmer for many discussions about this and related issues.

Our paper can be further related to three other strands of the literature. Many authors have recently inquired into the quantitative fit for the upper-tail of the wealth distribution. Nirei and Aoki (2016) construct a neoclassical growth model that yields a Pareto distribution for the upper tail.[14] They work with closed-form solutions in the absence of labour income risk. When there is labour income risk, they analyse a stationary economy. Similar to our work, Aoki and Nirei (2017) also describe the dynamics of distributions by employing FPEs. However, due to the absence of stochastic labour income, they are able to obtain closed form solutions, such as Angeletos (2007). Cao and Luo (2017) allow for stochastic returns and for ex-ante heterogeneity in labour productivity in a growth model. They also have a closed-form solution for policy functions that enables them to study transitional paths of the effects of policy reforms on top end wealth inequality and welfare.

Until recently, only a few papers have employed FPEs (i.e. forward Kolmogorov equations) in their analysis. Bayer and Wälde (2010a, sect. 5) showed how to derive them for relatively general cases (using a Bewley-Huggett-Aiyagari model as example).[15] More recently, FPEs became much more popular and we share the belief in their usefulness with Benhabib, Bisin and Zhu (2016), Achdou et al. (2020), Jones and Kim (2017), Cao and Luo (2017), Aoki and Nirei (2017), Kaplan et al. (2018)[16] and Nuño and Moll (2018).[17] We contribute to this literature by enquiring into the quantitative merits of FPEs. We match theoretical densities to empirical densities over their full range (hence, we do not focus on the upper tail or specific moments). We study in particular how the density of wealth evolves over time and how well such a model can replicate the empirical evolution of wealth densities. Due to the presence of jump processes and the implied linearity of the partial differential equations, we use the method of characteristics to solve them (see app. D.2). We also inquire into the quantitative success of the idiosyncratic interest rate hypothesis by comparing the mean and standard deviation of our densities to the mean and standard deviations reported in the literature (e.g. Flavin and Yamashita, 2002, Fagereng et al., 2018). As mentioned previously, we show that this risky-return approach is quantitatively more than successful.[18]

[14] See Gabaix (2009) for an excellent introduction to Power law/ Pareto distributions.

[15] Bayer and Wälde (2015, p. 4) provide a short survey on the use of FPEs in economics.

[16] From a quantitative perspective, Kaplan et al (2018, table 5) target moments of the wealth distribution and match top shares in a New Keynesian model with heterogenous agents. Their quantitative analysis focuses on stationary distributions and they abstract from capital income risk.

[17] Achdou et al. (2014) provide an overview of partial differential equation models in macroeconomics. Ahn et al. (2017) describe numerical methods for continuous time models. These methods are faster and more accurate than standard methods and allow to solve larger models as well.

[18] Kasa and Lei (2018) study a Blanchard-Yaari model with Knightian uncertainty where individuals are uncertain about the true model - as opposed to (standard) models with risk. Modelling capital income risk by Brownian motion, they also employ Fokker-Planck equations. Our wealth distributions are quantitatively closer to data than theirs. By contrast, they do not require type- and scale-dependence to obtain sufficiently

Our labour income process is inspired by the search and matching literature starting with Diamond (1982), Mortensen (1982) and Pissarides (1985).[19] We let labour income fluctuate between a wage when employed and unemployment benefits when unemployed. Corresponding transition rates are quantified by average durations in employment and unemployment, respectively, in the NLSY. We agree that any realistic income process would need much more structure (see e.g. Blundell et al., 2015 or Guvenen et al., 2019). An outstanding example for an empirically more convincing income process is the precautionary saving and on-the-job search model by Lise (2013). Interestingly, his analysis does not require a "superstar" state to obtain a satisfactory fit for one cross-section of wealth at a constant interest rate. Yet, he treats all workers within one education class as being identical. This creates low probability events such that workers in the state with the highest observed wage within this education class have a very strong saving motive. An argument in favour of our simple income structure is the well-known finding that the empirical skewness in the earnings distribution is not enough to generate sufficiently skewed and thick-tailed wealth distributions (Benhabib and Bisin, 2018, sect. 3.1). Therefore, we demonstrate that even with such a simple process, we can match the dynamics of the distribution of wealth.[20] We also show below that two (instantaneous) income states nevertheless imply a continuous monthly or annual income distribution.

[Table of contents] The next section describes an individual facing idiosyncratic risk resulting from a stationary interest rate process and from idiosyncratic risk resulting from labour income with (a deterministic) trend. Given this growth process, section 3 derives a stationary representation of our dynamic economy and defines equilibrium. Section 4 analyses the dynamics of distributions for detrended variables and shows how distributions evolve for variables with trend. Section 5 demonstrates the empirical fit of the evolution of the distributions for wealth (with trend). Finally, section 6 concludes.

2 The model

2.1 The individual

Our individual owns wealth $a(t)$ that increases in a deterministic fashion when capital income $r(t)a(t)$ plus labour income $z(t)$ exceeds consumption $c(t)$,

$$da(t) = \{r(t)a(t) + z(t) - c(t)\}dt. \tag{1}$$

fast increases in inequality.

[19] Recent contributions include Piyapromdee (2018), Shepard (2017) or Launov and Wälde (2013).

[20] In a dynamic wealth-inequality accounting analysis, one would ask how changes of the wealth distribution over time can be attributed to the labour income process, to the distribution of capital returns, to the role of bequests, the fiscal system, medical expenses or entrepreneurial activity. De Nardi and Fella (2017) conclude that more data is needed to determine the relative importance of these potential factors.

The instantaneous interest rate is denoted by $r(t)$. It fluctuates between a low value r_{low} and a high value r_{high},

$$dr(t) = [r_{\text{high}} - r(t)]\, dq_{\text{low}}(t) + [r_{\text{low}} - r(t)]\, dq_{\text{high}}(t). \tag{2}$$

The arrival rates of the corresponding Poisson processes $q_{\text{low}}(t)$ and $q_{\text{high}}(t)$ are $\lambda^{\text{low}} > 0$ and $\lambda^{\text{high}} > 0$, respectively. The interest rate jumps from its current level $r(t)$ to the new level r_{low} or r_{high} when the corresponding increment, $dq_{\text{high}}(t)$ or $dq_{\text{low}}(t)$, equals unity.

The arrival rates λ^{low} and λ^{high} are heterogenous across individuals. This captures the idea that individuals differ in their financial ability i. Each individual draws arrival rates from a two-dimensional distribution before becoming economically active. Once drawn, the arrival rates remain constant throughout life. When an individual draws a high λ^{low}, it leaves the state with a low return relatively quickly. When the individual has a high λ^{high}, it leaves the state with high returns relatively quickly. To make the model parsimonious, we make λ^{high} a falling function of λ^{low}. This makes sure that an individual that draws a high λ^{low} has a low λ^{high}. The individual with a high λ^{low} will therefore spend more time in expectation in the regime with the high return than in the regime with the low return. A high λ^{low} therefore stands for a high financial ability. To fix ideas, let us assume there are n different financial types i, i.e. n different levels λ_i^{low} (and therefore n implied levels λ_i^{high}) from which the individual draws. The probability to be an individual of financial type i is denoted by p_i.

Labour income $z(t)$ fluctuates between two paths, $z(t) \in \{w(\Gamma(t)), b(\Gamma(t))\}$. The paths are given by the wage path $w(\Gamma(t)) = \hat{w}\Gamma(t)$ while working and unemployment benefit path $b(\Gamma(t)) = \hat{b}\Gamma(t)$ while unemployed.[21] Initial income levels at $t = 0$ are denoted by $\hat{w} > \hat{b} > 0$. The underlying trend component $\Gamma(t)$ follows

$$\Gamma(t) \equiv \Gamma_0 e^{gt} \tag{3}$$

and can be imagined to result from technological progress with initial level Γ_0. Labour income therefore grows at a constant rate g with occasional jumps. Formally, labour income $z(t)$ follows

$$dz(t) = gz(t)\, dt + [w(\Gamma(t)) - z(t)]\, dq_\mu(t) + [b(\Gamma(t)) - z(t)]\, dq_s(t). \tag{4}$$

The growth process is visible in the deterministic dt-part of this equation. The jumps are described by the increments $dq_i(t)$, $i \in \{s, \mu\}$, of Poisson processes with constant arrival rates $\mu > 0$ moving the individual from unemployment to employment and $s > 0$ moving the individual from employment to unemployment.[22]

[21] Labour income as well as interest rates are understood as after-tax values. The effect of tax changes, which are central to Kaymak and Poschke (2016) or Hubmer et al. (2019), are therefore implicitly covered in our net wage and net interest rate paths.

[22] We emphasize that this process implies a *continuous* income distribution for e.g. monthly or annual income. See footnote 42 for details.

Initial conditions for these three differential equations (1), (2) and (4) are random. Wealth of an employed worker and wealth for an unemployed worker are given by two independent densities (that could be degenerate).[23] The initial value $r(0)$ is drawn from a distribution with realizations $\{r_{\text{low}}, r_{\text{high}}\}$ and the individual is allocated to the state of being employed with a certain probability.[24]

There is a long and fruitful tradition in macroeconomics where trends are filtered out of the data before comparing model predictions with data. We allow for a growth component in our model and compare the model predictions with data directly for two reasons. First, we would like to understand the evolution of wealth distributions over time and it seemed more natural and more parsimonious not to detrend changing wealth distributions. Second, our model is then closer to the discussions surrounding Piketty's "$r > g$" hypothesis.[25]

Individuals maximize their expected present value of their utility streams. Given a time preference rate ρ, their intertemporal utility $U(0)$ reads

$$U(0) = E \int_0^\infty e^{-\rho t} u(c(t))\, dt. \tag{5}$$

Instantaneous utility $u(c(t))$ is a function of consumption and is assumed to reflect constant relative risk aversion (CRRA) with a risk aversion parameter σ,

$$u(c(t)) = \begin{cases} \frac{c(t)^{1-\sigma}-1}{1-\sigma}, & \sigma > 0, \sigma \neq 1, \\ \ln(c(t)), & \sigma = 1. \end{cases} \tag{6}$$

We let the individual choose optimal consumption as a function of the wealth level $a(t)$, labour income $z(t)$ and the interest rate $r(t)$, $c(t) \equiv c(a(t), z(t), r(t))$.

Individuals can borrow up to their natural borrowing limit. It is given by the amount of debt that they can pay back with probability one, i.e. in all possible states of the world. We assume that all debt contracts are such that the current personal interest is paid (in the case of positive wealth) or has to be paid (in the case of debt). The worst possible state for a person in debt is therefore the high interest rate r_{high}. We further require consumption to be sufficiently large to guarantee survival of the individual, i.e. $c^z(a) \geq c^{\min}$. For notational simplicity, we

[23] While a density for initial wealth sounds unusual, it becomes very plausible when thinking of durable consumption goods or assets like e.g. a bike, a car or a house. The price of these goods is not easily quantified at a high precision.

[24] In our calibrated version, the probability for an individual to be unemployed in 1986 is set equal to the empirical unemployment rate in 1986. The product of the unemployment rate and the empirical wealth density for the unemployed is the initial condition for the wealth density of the unemployed. The initial condition for the employed is the empirical wealth density for the employed times one minus the unemployment rate. The initial interest rate $r(0)$ can be either r_{low} or r_{high} with a probability of p_0 and $1 - p_0$.

[25] See e.g. Piketty (2015a,b), Mankiw (2015), Jones (2015) or Hubmer et al. (2019). We briefly discuss our findings in this respect in footnote 34.

relate this minimum consumption level to unemployment benefits by

$$c^{\min}(t) = \xi b(t) \tag{7}$$

where $0 < \xi < 1$ measures the amount of unemployment benefit needed to survive.[26] As a consequence, a share $1-\xi$ of unemployment benefits can be used to pay interests on debt. This implies (see app. A.1) that the natural borrowing limit is

$$a^{\text{nat}}(t) = -\frac{(1-\xi) b(t)}{r_{\text{high}} - g}. \tag{8}$$

The debt level is the higher, the larger the growth rate g of unemployment benefits. It falls when the minimum consumption level requires a larger share ξ of unemployment benefits and when the interest rate r rises.[27] As our natural borrowing limit is negative, we explicitly allow for debt in our model. We assume that individuals are charged the same interest rate on debt as they earn on positive wealth levels.[28]

2.2 Keynes-Ramsey rules

Optimal consumption is a function $c(a(t), z(t), r(t))$. To simplify notation, we write this in short hand as $c^{z(t)}_{r(t)}(a(t))$ and, if possible, we will suppress time arguments. When our individual maximizes utility, they take the current wealth level, the uncertainty from labour income growth, the current technological level and the interest rate *level* into account. Individuals are assumed to be myopic with respect to interest rate *changes*. Changes in individual returns come as a surprise and are not anticipated.[29] Optimal consumption for an employed worker is described by a Keynes-Ramsey rule that reads (see app. A.2)[30]

$$dc^w_r(a) = \frac{c^w_r(a)}{\sigma}\left\{r - \rho + s\left[\left(\frac{c^w_r(a)}{c^b_r(a)}\right)^{\sigma} - 1\right]\right\} dt + \left[c^b_r(a) - c^w_r(a)\right] dq_s. \tag{9a}$$

[26] We introduce this strictly positive minimum consumption level out of plausibility and as ξ allows us to adjust the left tail of the theoretical wealth distribution in our calibration.

[27] The borrowing limit would not exist (it would be minus infinity) if the growth rate g is higher than the interest rate r_{high}. In our quantitative analysis below, $r_{\text{high}} > g$ holds.

[28] One could think about more elaborate setups with explicit debt contracts. We follow the same idea for negative wealth as for positive wealth: individuals differ in their luck when making investments. They also differ in their luck when borrowing resources.

[29] This assumption allows us to work with a two-dimensional system. With anticipation of uncertain interest rates, we would have four (coupled) Keynes-Ramsey rules and a system of four (coupled) Fokker-Planck equations. While this is theoretically straightforward (and numerically only requires more code), we stick here in this first quantitative application of Fokker-Planck equations in economics to this two-dimensional system. Future work with anticipation of interest rate changes could actually reduce the dimensionality if one allowed for a continuum of wage incomes (instead of our discrete labour income distribution) as in Lise (2013).

[30] This rule is very similar to the optimal consumption rule in Lise (2013) or in Bayer and Wälde (2010b).

To understand this stochastic differential equation, consider first the case of employment as an absorbing state, i.e. the case of a separation rate s of zero. Consumption grows when the interest rate r exceeds the time preference rate ρ. With a positive arrival rate s, we see that consumption grows faster: The term in squared brackets in the deterministic part is positive as consumption when employed at a wage w is larger than when unemployed when receiving benefits $b < w$ (which implies higher marginal utility from consumption when unemployed). As a consequence, consumption growth tends to be faster. This is obviously the effect of precautionary saving. As individuals anticipate the risk of experiencing lower labour income, they reduce the consumption level, accumulate wealth faster and thereby experience faster consumption growth. The jump term says that consumption jumps from its optimal level $c_r^w(a)$ to the level $c_r^b(a)$ when the worker loses their job.

For the unemployed worker, the Keynes-Ramsey rule reads

$$dc_r^b(a) = \frac{c_r^b(a)}{\sigma}\left\{r - \rho - \mu\left[1 - \left(\frac{c_r^b(a)}{c_r^w(a)}\right)^{\sigma}\right]\right\}dt + \left[c_r^w(a) - c_r^b(a)\right]dq_\mu. \tag{9b}$$

Here, the effect of labour income uncertainty is reversed. Again, the term in squared brackets of the deterministic part is positive such that overall consumption growth is smaller as compared to a situation where unemployment lasts forever (i.e. when $\mu = 0$). Individuals anticipate that at some point in the future labour income will be high again such that they increase their consumption level and thereby save less. This could be called "post-cautionary dissaving".

3 Detrending and equilibrium

3.1 Detrending

Before we can define our solution concept for the individual's maximization problem, we derive a stationary version of the model. Based on the trend (3), we define detrended variables,

$$\hat{z}(t) \equiv \frac{z(t)}{\Gamma(t)},\ \hat{a}(t) \equiv \frac{a(t)}{\Gamma(t)},\ \hat{c}_r^{\hat{z}}(\hat{a}(t)) \equiv \frac{c_r^z(a(t), \Gamma(t))}{\Gamma(t)}, \tag{10}$$

which evolve over time as well. Using the laws of motion for the underlying variables, the detrended income process follows (see app. B.1)

$$d\hat{z}(t) = (\hat{w} - \hat{z})\,dq_\mu(t) + \left(\hat{b} - \hat{z}\right)dq_s(t). \tag{11}$$

Detrended labour income is therefore either $\hat{w}$ or $\hat{b}$, consistent with the construction of labour income before (3). The Poisson processes and arrival rates in (11) are identical to those used in the version with trend in (2).

The evolution of detrended wealth $\hat{a}(t)$ reads

$$d\hat{a}_r^{\hat{z}}(t) = \left\{(r(t) - g)\,\hat{a}_r^{\hat{z}}(t) + \hat{z}(t) - \hat{c}_r^{\hat{z}}(\hat{a}(t))\right\}dt. \tag{12}$$

The initial densities for $\hat{a}_r^{\hat{z}}(t)$ are given by the one from above for (1) as the trend (3) sets in only at $t = 0$ such that the distribution for $a_r^z(0)$ and $\hat{a}_r^{\hat{z}}(0)$ are the same.

We can express the evolution of detrended consumption as a function of detrended wealth for the time in between transitions on the labour market and for a given interest rate $r \in \{r_{\text{low}}, r_{\text{high}}\}$ as (see app. B.1)

$$\frac{d\hat{c}_r^{\hat{w}}(\hat{a})}{d\hat{a}} = \frac{\frac{r-\rho}{\sigma} - g + \frac{s}{\sigma}\left[\left(\frac{\hat{c}_r^{\hat{w}}(\hat{a})}{\hat{c}_r^{\hat{b}}(\hat{a})}\right)^{\sigma} - 1\right]}{(r-g)\hat{a} + \hat{w} - \hat{c}_r^{\hat{w}}(\hat{a})}\hat{c}_r^{\hat{w}}(\hat{a}), \tag{13a}$$

$$\frac{d\hat{c}_r^{\hat{b}}(\hat{a})}{d\hat{a}} = \frac{\frac{r-\rho}{\sigma} - g - \frac{\mu}{\sigma}\left[1 - \left(\frac{\hat{c}_r^{\hat{b}}(\hat{a})}{\hat{c}_r^{\hat{w}}(\hat{a})}\right)^{\sigma}\right]}{(r-g)\hat{a} + \hat{b} - \hat{c}_r^{\hat{b}}(\hat{a})}\hat{c}_r^{\hat{b}}(\hat{a}). \tag{13b}$$

Finally, the detrended natural borrowing limit with the corresponding minimum consumption level can be obtained from (7) and (8) as

$$\hat{c}^{\min} = \xi\hat{b}, \quad \hat{a}^{\text{nat}} = \frac{a^{\text{nat}}(t)}{\Gamma(t)} = -\frac{(1-\xi)\hat{b}}{r_{\text{high}} - g}. \tag{14}$$

Equations (11) to (14) form the basis of our analysis of the dynamics of (detrended) distributions and of our definition of optimal behaviour. These dynamics are conditional on the interest rate $r \in \{r_{\text{low}}, r_{\text{high}}\}$. Before we define optimal behaviour, let us gain some intuition for the distribution of wealth – which crucially depends on r.

3.2 Consumption and wealth dynamics

To understand the dynamics of consumption and wealth, it is crucial to distinguish between three "regimes". They are determined by the level of the interest rate relative to preference, growth and job-market parameters. Our individual finds themself in the low-interest-rate regime when

$$r < \rho + \sigma g. \tag{15}$$

This condition follows from analysing the Keynes-Ramsey rule (9a) of the employed worker (see app. B.2). This is the regime the precautionary savings literature has looked at so far: Individuals have an incentive to save because of precautionary motives. At the same time, they have an incentive to dis-save as returns r to wealth are lower than the time preference rate ρ (adjusted here for the growth rate as we allow for growth of labour income).

Our individual finds herself in the high-interest-rate regime when the interest rate satisfies

$$\rho + \sigma g < r < \rho + \sigma g + \mu. \tag{16}$$

This condition follows from analysing the optimal consumption rules (9) for both the employed and the unemployed worker. For this regime, the dis-saving motive is no longer present.

Finally, there is a "very-high-interest-rate" regime when r exceeds $\rho + \sigma g + \mu$.[31] We do not consider this regime to be of empirically relevance: The arrival rate μ for jobs is of an order of magnitude larger than any real world interest rate (see tab. 1 below). As a consequence, we do not expect that individual interest rates r are persistently larger than $\rho + \sigma g + \mu$. The following analysis will therefore employ the low-interest-rate and the high-interest-rate regime.[32]

- Low-interest-rate regime

For the low-interest-rate regime from (15), consumption and wealth dynamics can be illustrated in fig. 1. The figure depicts wealth $\hat{a}$ on the horizontal and consumption $\hat{c}^{\hat{w}}_{r_{\text{low}}}(\hat{a})$ on the vertical axis. There are two zero-motion lines for wealth (for the employed and for the unemployed worker) and one zero-motion line for consumption of employed workers. Consumption for unemployed workers falls for any wealth-consumption pair. The implied arrow-pairs indicating the changes in wealth and consumption over time are also drawn.

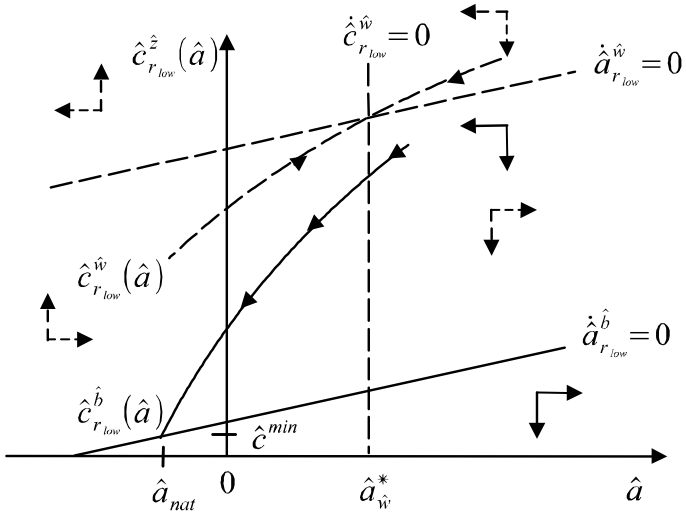

Figure 1 *Consumption and wealth dynamics in the low-interest rate regime (15)*

Employed workers experience rising (detrended) consumption $\hat{c}^{\hat{w}}_{r_{\text{low}}}(\hat{a})$ over time for $\hat{a} < \hat{a}^{*}_{\hat{w}}$, i.e. as long as they are sufficiently poor. This is the dashed trajectory drawn in fig. 1. Detrended consumption $\hat{c}^{\hat{b}}_{r_{\text{low}}}(\hat{a})$ of unemployed workers always falls, as the solid trajectory illustrates. As a consequence, wealth is constrained between a lower bound from (14) and an upper bound $\hat{a}^{*}_{\hat{w}}$.[33]

[31] We ignore the singular cases where r lies on the boundaries for brevity.

[32] These conditions illustrate that one could obtain similar quantitative findings for a stochastic time preference rate (as in Krusell and Smith, 1998) and a fixed interest rate. Stochastic interest rates have the advantage that they can be observed more easily than stochastic time preference rates. This allows to test the model predictions more easily as we do in sect. 5.4.

[33] See app. B.2 for a more formal background and the web appendix of Lise (2013) for a similar illustration. Lise does not look at exploding regimes and the evolution of wealth over time.

As the figure shows, there is a (temporary) steady state (TSS) at the upper end $\hat{a}^*_{\hat{w}}$. In this TSS where $d\hat{a}/dt = 0$ the consumption level in the state of employment satisfies

$$\hat{c}^{\hat{w}}_{r_{\text{low}}}\left(\hat{a}^*_{\hat{w}}\right) = r_{\text{low}}\hat{a}^*_{\hat{w}} + \hat{w}. \tag{17}$$

The steady state is called temporary as any employed worker will eventually be hit by an unemployment shock. Consumption then drops according to the following relative consumption level

$$\frac{\hat{c}^{\hat{b}}_{r_{\text{low}}}\left(\hat{a}^*_{\hat{w}}\right)}{\hat{c}^{\hat{w}}_{r_{\text{low}}}\left(\hat{a}^*_{\hat{w}}\right)} = \left(1 - \frac{r_{\text{low}} - \rho - \sigma g}{s}\right)^{-1/\sigma}. \tag{18}$$

This ratio also follows from (see again app. B.2) studying the Keynes-Ramsey rule (13a) of the employed worker.

- High-interest-rate regime

When the interest rate is at a high level as in (16), consumption $\hat{c}^{\hat{w}}_{r_{\text{high}}}\left(\hat{a}\right)$ of employed workers rises for any wealth level. This is the dashed trajectory depicted in figure 2. Just as in fig. 1, wealth is plotted on the horizontal and consumption on the vertical axis. There are now two zero-motion lines for the state of unemployment and only one (for wealth) for the employment state. Consumption $\hat{c}^{\hat{b}}_{r}\left(\hat{a}\right)$ of unemployed workers rises only for $\hat{a} > \hat{a}^*_{\hat{b}}$, i.e. if the unemployed worker is sufficiently rich (see app. B.3 for details). This is the solid trajectory of figure 2.

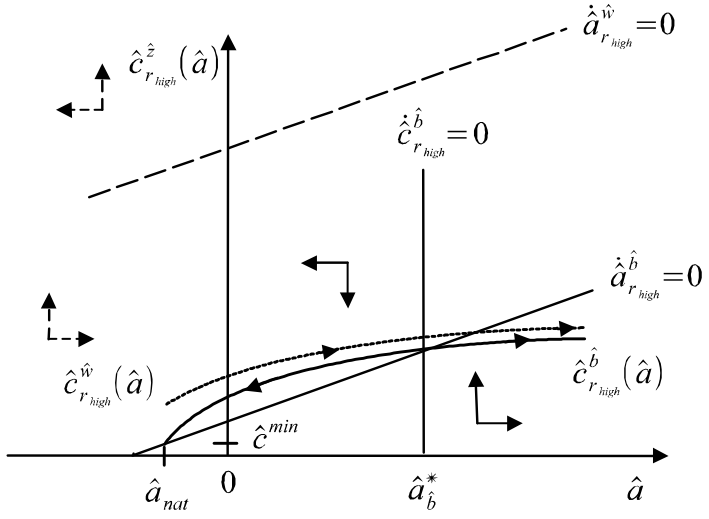

Figure 2 *Consumption and wealth dynamics in the high-interest-rate regime (16)*

The consumption level at the temporary steady state for the high interest rate is given by

$$\hat{c}^{\hat{b}}_{r_{\text{high}}}\left(\hat{a}^*_{\hat{b}}\right) = r_{\text{high}}\hat{a}^*_{\hat{b}} + \hat{b} \tag{19}$$

and relative consumption is given by (see app. B.3)

$$\frac{\hat{c}^{\hat{b}}_{r_{\text{high}}}\left(\hat{a}^*_{\hat{b}}\right)}{\hat{c}^{\hat{w}}_{r_{\text{high}}}\left(\hat{a}^*_{\hat{b}}\right)} = \left(1 - \frac{r_{\text{high}} - \rho - \sigma g}{\mu}\right)^{1/\sigma}. \tag{20}$$

The natural borrowing limit and the minimum consumption level remain unchanged as in (14). Note that there is no saddle-path property of the consumption path $\hat{c}^{\hat{b}}_{r_{\text{high}}}(\hat{a})$ in this regime. As we consider $d\hat{c}^{\hat{b}}_{r_{\text{high}}}(\hat{a})/d\hat{a} > 0$ to be empirically convincing, we impose that $\hat{c}^{\hat{b}}_{r_{\text{high}}}(\hat{a})$ crosses the steady state $\left(\hat{a}^*_{\hat{b}}, \hat{c}^{\hat{b}}_{r_{\text{high}}}\left(\hat{a}^*_{\hat{b}}\right)\right)$, as drawn.

- Consumption and wealth distributions

We are now in a position to gain some intuition about the distribution of wealth and consumption in our model. When the interest rate is in the low-interest rate regime, the wealth level of an individual is bounded, at least after some finite transition period, between $\hat{a}^{\text{nat}}$ and $\hat{a}^*_{\hat{w}}$. This follows from the trajectories drawn in fig. 1. With an initial distribution of wealth between $\hat{a}^{\text{nat}}$ and any wealth level lower than or equal to $\hat{a}^*_{\hat{w}}$, there would be a density of wealth within this support for any future point in time.

When the interest rate is in the high regime, as illustrated in fig. 2, there would be no upper bound and a stationary long-run wealth distribution would not exist. With an initial distribution of wealth between $\hat{a}^{\text{nat}}$ and $\hat{a}^*_{\hat{b}}$, the right tail of the density of wealth for each future point in time shifts to the right due to those who are employed. If initial wealth is distributed between $\hat{a}^{\text{nat}}$ and a wealth level higher than $\hat{a}^*_{\hat{b}}$, the right tail of the density of wealth shifts to the right due to both the employed and the unemployed workers.

In our setup, the interest rate follows (2) and takes values both in the low and in the high regime. Wealth will evolve "normally" for $r < \rho + \sigma g$ and remain bounded from above (and below). Wealth will be accumulated very quickly when in the high regime from (16). The latter is the basic mechanism through which we obtain a wealth distribution with enough probability mass in the right tail. This regime is called "explosive wealth accumulation" by Benhabib and Bisin (2018).[34] The support of wealth we work with in our detrended model is therefore given by

$$\hat{a} \in \left[\hat{a}^{\text{nat}}, \hat{a}^{\max}\right]. \tag{21}$$

The lower limit $\hat{a}^{\text{nat}}$ is from (14), the upper limit $\hat{a}^{\max}$ of our wealth variable is determined further below such that the model density with trend in (24) covers all empirical wealth observations (see also footnote 47). For the time being, it is a large number. We know that this $\hat{a}^{\max}$ is a finite number as we study individuals only over a finite length of time (i.e. 22 years).

3.3 Optimal behaviour

We now describe optimal behaviour. An individual behaves optimally when following the Keynes-Ramsey rules (13a) when employed and (13b) when unemployed. Consumption jumps

[34] Conditions (15) and (16), distinguishing between a "normal" and an "explosive" regime provide an extension of the $r > g$ condition that can be derived from steady state analyses based on Pareto-distributions (Piketty and Zucman, 2015, ch. 15.5.4).

from one equilibrium path to the other when the individual loses or finds a job, as shown in (9).

In the low-interest-rate regime, the two boundary conditions for the two Keynes-Ramsey rules are given by the consumption level $\hat{c}^{\hat{w}}_{\text{low}}(\hat{a}^*_{\hat{w}})$ from (17) and the consumption level $\hat{c}^{\hat{b}}_{\text{low}}(\hat{a}^*_{\hat{w}})$ following from (18) at the temporary steady state. The wealth level $\hat{a}^*_{\hat{w}}$ at the temporary steady state is defined such that consumption in the state of unemployment at the natural borrowing limit is given by the minimum consumption level, i.e. $\hat{c}^{\hat{b}}_{\text{low}}(\hat{a}^{\text{nat}}) = \hat{c}^{\min}$.[35]

When the interest rate jumps to the high regime, the boundary conditions at the temporary steady state change to the values $\hat{c}^{\hat{b}}_{\text{high}}\left(\hat{a}^*_{\hat{b}}\right)$ from (19) and the consumption level $\hat{c}^{\hat{w}}_{\text{high}}\left(\hat{a}^*_{\hat{b}}\right)$ following from (20). The wealth level $\hat{a}^*_{\hat{w}}$ for the zero-motion line of consumption is replaced by $\hat{a}^*_{\hat{b}}$, where the latter is determined according to the same logic, i.e. such that $\hat{c}^{\hat{b}}_{\text{high}}(\hat{a}^{\text{nat}}) = \hat{c}^{\min}$.

4 Distributional dynamics

Having understood the dynamics of consumption and wealth qualitatively for a realization of uncertainty, we can now study the distribution of wealth more generally.

4.1 Densities and subdensities of wealth

We start by studying the joint distribution of detrended wealth $\hat{a}(t)$ and income $\hat{z}(t)$ as governed by (12) and (11), given optimal consumption $\hat{c}^{\hat{z}}_r(\hat{a}(t))$ as just defined. We denote this joint density for a point t in time by $p^{\hat{z}}(\hat{a}, t)$. The density is continuous in wealth $\hat{a}$ and discrete in labour income $\hat{z}$. For the time being, we describe densities for a given and constant interest rate r. We explain further below how changes in the interest rate are taken into account. Given the discrete nature of $\hat{z}$, the joint density can be split into two "sub-densities" $p^{\hat{w}}(\hat{a}, t)$ and $p^{\hat{b}}(\hat{a}, t)$.[36] The density of wealth $p(\hat{a}, t)$ is then

$$p(\hat{a}, t) = p^{\hat{w}}(\hat{a}, t) + p^{\hat{b}}(\hat{a}, t). \tag{22}$$

[35] This also illustrates the idea behind the numerical solution: Guess $\hat{a}^*_{\hat{w}}$ and check whether $\hat{c}^{\hat{b}}(\hat{a}^{\text{nat}}) = \hat{c}^{\min}$ holds. If not, adjust the guess.

[36] Integrating the sub-density $p^{\hat{b}}(\hat{a}, t)$ over the range of wealth gives the unemployment rate at t.

The dynamics of the subdensities $p^{\hat{w}}(\hat{a},t)$ and $p^{\hat{b}}(\hat{a},t)$ is governed by the Fokker-Planck equations (FPEs). They read[37]

$$\frac{\partial}{\partial t}p^{\hat{w}}(\hat{a},t)+\left[(r-g)\hat{a}+\hat{w}-\hat{c}_r^{\hat{w}}(\hat{a})\right]\frac{\partial}{\partial \hat{a}}p^{\hat{w}}(\hat{a},t)=\left[\frac{d\hat{c}_r^{\hat{w}}(\hat{a})}{d\hat{a}}-(r-g)-s\right]p^{\hat{w}}(\hat{a},t)+\mu p^{\hat{b}}(\hat{a},t), \tag{23a}$$

$$\frac{\partial}{\partial t}p^{\hat{b}}(\hat{a},t)+\left[(r-g)\hat{a}+\hat{b}-\hat{c}_r^{\hat{b}}(\hat{a})\right]\frac{\partial}{\partial \hat{a}}p^{\hat{b}}(\hat{a},t)=sp^{\hat{w}}(\hat{a},t)+\left[\frac{d\hat{c}_r^{\hat{b}}(\hat{a})}{d\hat{a}}-(r-g)-\mu\right]p^{\hat{b}}(\hat{a},t), \tag{23b}$$

These two equations constitute a system of two coupled partial differential equations. The partial derivatives with respect to t and $\hat{a}$ describe the time evolution and the cross-sectional dimension of the density of wealth, respectively. The evolution of the wealth density is directly linked to optimal consumption-saving paths as the FPEs display optimal consumption levels $\hat{c}^{\hat{z}}(\hat{a})$ and their derivatives. When the wealth distribution has reached its stationary distribution, the partial differential equations simplify to a set of ordinary differential equations in wealth.[38]

Note that this approach does not work with and does not require stationary distributions. The equations rather describe the evolution of the distribution (which might converge to a stationary distribution).[39] We start at some (empirically) given initial distribution and then compute the changes of the distribution over a certain length of time (see app. D.2 for more details).

4.2 Distributional dynamics of variables with trend

We have now obtained (i) the policy functions resulting from (13a) and (13b) for the two interest rate regimes and (ii) the densities and their evolution over time from (23a) and (23b). In a third step, we need to transform these findings for detrended variables back into *levels* before we can compare them with data. Going back to levels for "normal" variables is straightforward by inversion of (10), $z(t)=\hat{z}(t)\Gamma(t)$, $c_r^z(a)=\hat{c}_r^{\hat{z}}(\hat{a})\Gamma(t)$ and $a(t)=\hat{a}(t)\Gamma(t)$. The densities $p^{\hat{z}}(\hat{a},t)$ and $p(\hat{a},t)$ can be retransformed by Edgeworth's method of translation (Benhabib and Bisin, 2018, sect. 1.2, Wackerly et al., 2008, ch. 6.4, Wälde, 2012, theorem 7.3.2). This translation describes the link between a random variable ($\hat{a}$ in our case) and its transformation

[37] The derivation follows the approach described in Bayer and Wälde (2010a, sect. 5).

[38] Describing distributions by differential equations has a long tradition and goes back to the work of Karl Pearson in the 19th century. See Johnson et al. (1994, ch. 4) for an overview.

[39] In a setup with $g=0$ and $r<\rho$, Bayer et al. (2019) prove that a unique stationary distribution exists and is stable. The theoretical analysis by Benhabib et al. (2015) employs an exploding regime as well to obtain the fat right tail. The interest rate distribution in their general equilibrium model makes sure that overall, their model displays a stationary distribution of wealth.

($a(t) = \hat{a}(t)\Gamma(t)$ here). For our support (21) for detrended wealth $\hat{a}(t)$ and using trend (3), this transformation implies a support for wealth $a(t)$ that evolves over time t,

$$a(t) \in \left[\hat{a}^{\text{nat}}\Gamma(t), \hat{a}^{\max}\Gamma(t)\right]. \tag{24}$$

The density $g(a,t)$ of wealth with trend is then given by (see app. C.2.2)

$$g(a,t) = \frac{p(a(t)/\Gamma(t), t)}{\Gamma(t)}. \tag{25}$$

5 The empirical fit

Let us now turn to the main objective of this paper – to explore how risky returns can help to understand the evolution of wealth of the 1979 NLSY cohort over time. The importance of risky returns have also been studied by Benhabib, Bisin and Zhu (2011, sect. 5) and Benhabib, Bisin and Luo (2017). We complement their findings by always starting from an initial density of wealth. Hence, even when we target only one year (in contrast to the entire path), we always ask how the *evolution* from our initial (empirically given) wealth distribution to the final wealth distribution can be understood.

5.1 Data and quantitative phase diagram

5.1.1 Some descriptive statistics

We extract the wealth distribution from the NLSY79 for all waves that provide information on wealth.[40] A visual impression of the fairly equal distribution of wealth when individuals are young in 1986 and the steady increase in the spread as the cohort becomes older is provided by the left figure in fig. 3. The spread increases as some individuals become poorer as they were initially and some become richer. Both the left and the right legs move outwards. The right figure shows the density as predicted by our model with the (close to perfect) fit for 2008. We will discuss this and other fits in detail below.

The NLSY data is also used for computing various parameters in our model. An overview of those parameters and also of exogenously fixed parameters is in table 1 below. The unit of time in our model is 1 year. Given that our model emphasizes transitions between employment and unemployment, we do not attempt to hit moments of income distributions. We rather match the average duration in employment and unemployment by the (annual) job arrival rate μ and the separation rate s (implying an average unemployment rate of 5.1%). The average (annual)

[40] We employ the NLSY variable "net worth", see Nagel (2013, ch. 6) for more background.

wage in 1986 is $\hat{w}$ and the (annual) growth rate of labour income is g.[41,42] All nominal values are expressed in prices of 2008. We infer unemployment benefits by assuming a replacement rate of 30%. This is a compromise between the higher statutory replacement rate and the fact that benefits are not paid forever in the US (but are done so in our model).[43] The share for consumption ξ is computed such that the natural borrowing limit (8) corresponds to the smallest (perceivable) wealth level in the data.

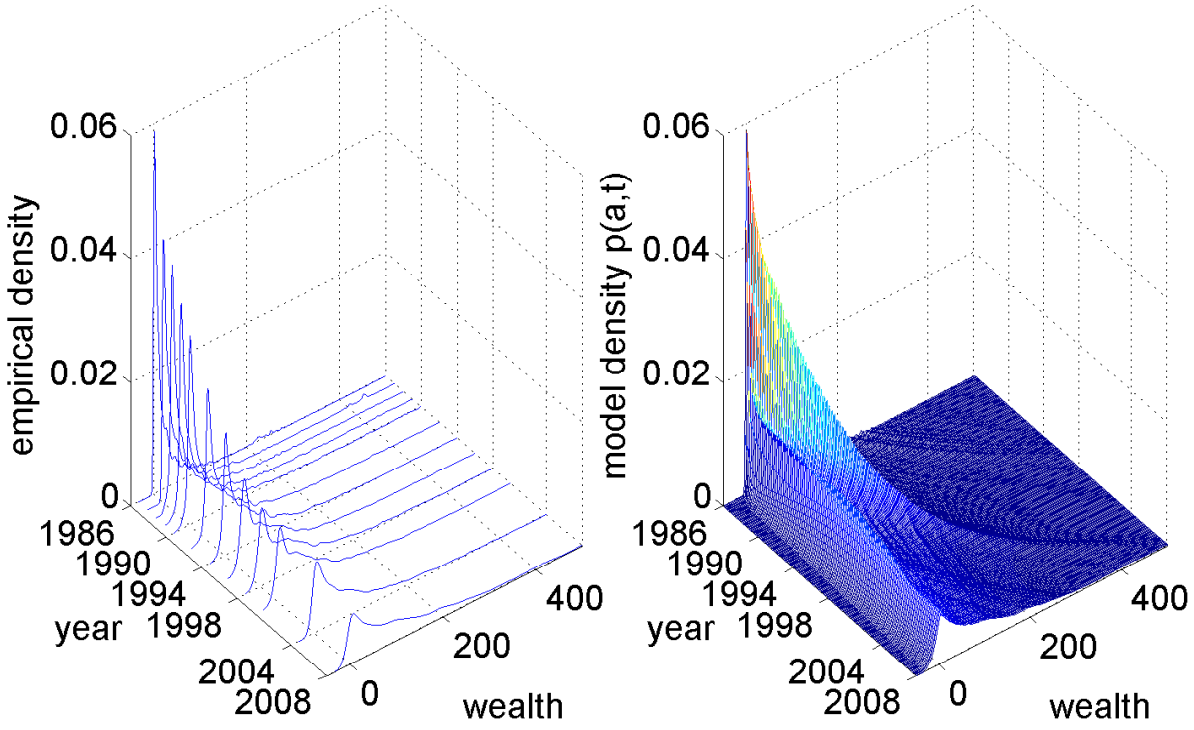

Figure 3 *The dynamics of the wealth distribution in the data and in the model (in 1000 US$ in prices of 2008)*

The time preference rate ρ and risk aversion σ are exogenously fixed and take standard values employed in many other calibrations. Robustness analyses are undertaken in section 5.3 below. The (instantaneous) interest rates r_{low} and r_{high} are also exogenously fixed. Their choice

[41] App. C.1.1 shows how we compute continuous-time wage rates and interest rates. The wage growth rate is so high as, inter alia, we look at a cohort whose average age is approx. 24 in 1986 and 46 in 2008.

[42] Obviously, individuals at each point in time earn one of two values (they are either employed or unemployed). Individuals nevertheless face a continuous annual income distribution. When we define annual income as $z^a(t) \equiv \int_{t-1}^{t} z(\tau)\, d\tau$ where $z(\tau)$ follows (4), we easily see that annual income is characterized by a continuous distribution. The mean of our annual income in 2008 is 53,943 US$ and thereby basically the same as the mean in the NLSY (53,471). Our standard deviation is considerably lower (4,218 US$) as compared to 50,679 in the data. As written in the introduction, we want to match the average wage and unemployment benefits in the tradition of Diamond-Mortensen-Pissarides models. Our labour income process therefore captures unemployment risk and not general labour income risk.

[43] Hall and Milgrom (2008, p. 11) briefly survey estimates in the literature of replacement rates. They consider 12% to be a lower bound and 36% to be an upper bound

was driven by two concerns. First, they must lie below and above the threshold level in (15), i.e. they must obey $r_{\text{low}} < \rho + \sigma g < r_{\text{high}}$. In all other cases, we would expect that the densities of wealth cannot be matched in a convincing way. Second, they should lie in a "reasonable range". The empirical evidence in sect. 5.4 confirms these values. Our model mean of 4.4% (annually) is in accordance with evidence by Flavin and Yamashita (2002), it is a bit high for Norwegian standards (Fagereng et al., 2018) and a bit lower than the findings in Cao and Luo (2017) or Moskowitz and Vissing-Jorgensen (2002). The robustness check in section 5.3 also includes the case of $r_{\text{high}} = 8\%$.

μ	s	$\hat{w}$	g	$\hat{b}/\hat{w}$	ξ	ρ	σ	r_{low}	r_{high}
264%	14%	27,330.8\$	3.4%	30%	97%	1%	1	3.5%	4.5%

Table 1 *Parameter values*

In contrast to Benhabib, Bisin and Zhu (2011) or Angeletos (2007), we cannot impose a standard deviation on our idiosyncratic interest rate process. While we do not estimate as in Benhabib, Bisin and Luo (2019), our calibration method implies a standard deviation that results from fitting wealth densities. Once we have matched wealth distributions in the best possible way, the implied standard deviation for the interest rate distribution will be compared with the empirical standard deviation (see sect. 5.4).

5.1.2 Quantitative phase diagram

Given the parameters in tab. 1, we can now plot a quantitative version of our qualitative phase diagrams in figures 1 and 2. Figure 4 displays the quantitative consumption paths for wealth levels between $\hat{a}^{\text{nat}}$ and $\hat{a}^*_{\hat{b}}$. The natural borrowing limit is $\hat{a}^{\text{nat}} = -\$16,341$. The (temporary) steady-state levels of wealth when the interest rate is low or high are $\hat{a}^*_{\hat{w}} = \$2,266$ and $\hat{a}^*_{\hat{b}} = \$930,132$, respectively (again in 2008 prices).

This figure is very instructive for understanding what the quantitative driving forces for the spread in wealth distributions are. First, for a given interest rate, the change in the employment status hardly has any impact on the consumption level. By contrast, for levels of wealth around $\hat{a}^*_{\hat{w}}$ or larger, an increase in the interest rate dramatically decreases the consumption level. Changes in the interest rate therefore have a much larger effect on the spread of wealth distributions than changes in labour income. Second, when the interest rate is low, the distribution of wealth converges to a range below $\hat{a}^*_{\hat{w}}$ which at $\$2,266$ is relatively low. The fat right tail of the distribution of wealth is therefore entirely driven by employed individuals that enjoy a high interest rate. This group experiences rising consumption and wealth levels. In fact, all other groups (the unemployed and those with low interest rate) experience consumption

and wealth levels that fall over time. The high-interest-rate regime, or the "exploding regime" in Benhabib and Bisin's (2018) terminology, is crucial for generating fat right tails of wealth distributions.

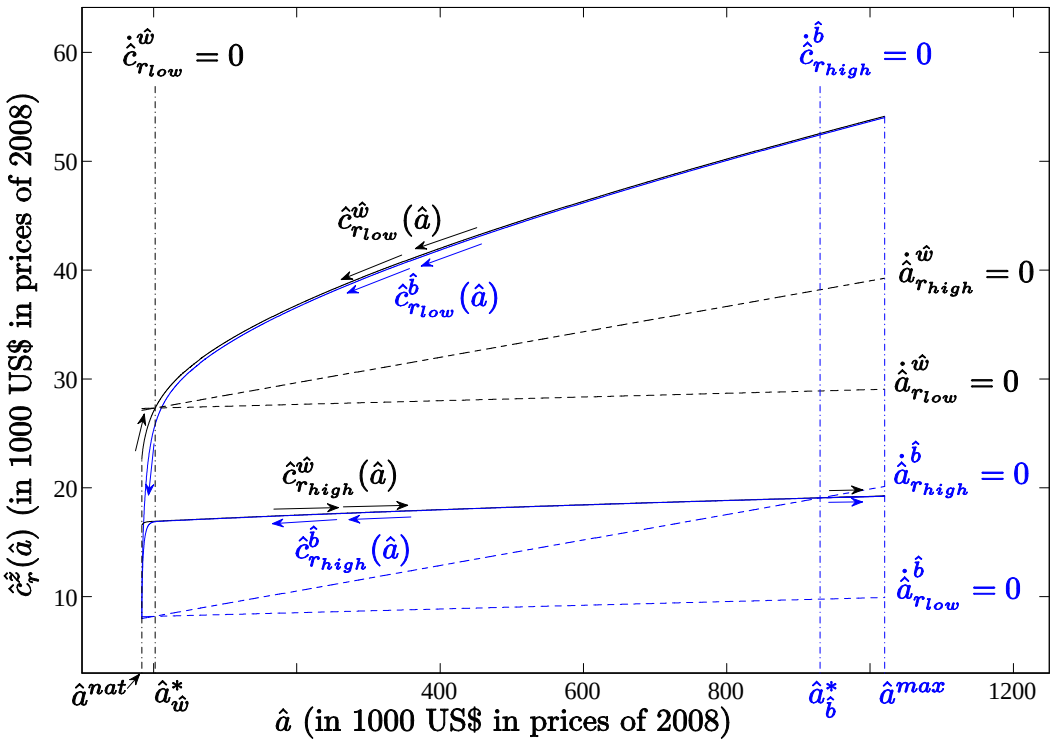

Figure 4 *Quantitative phase diagram*

Looking at the vertical axis of fig. 4 shows that consumption increases by around 50% at $\hat{a}^*_{\hat{w}}$ when the interest rate drops and more than doubles at $800.000US\$$ of wealth. How can this increase be understood? The closed-form solution for consumption in a deterministic optimal saving problem reads $c(t) = \frac{\rho-(1-\sigma)r}{\sigma}\left\{a(t) + \int_t^\infty e^{-r[\tau-t]}w(\tau)\,d\tau\right\}$ (see e.g. Wälde, 2012, ch. for a textbook derivation). When the interest rate rises, the effect via the consumption-propensity (the fraction) is ambiguous and depends on risk aversion σ. The quantitatively much larger negative effect comes through the fall in the present value of labour income (the integral) when the interest rate r rises. Even though we do not have a deterministic model, we believe that this is the main channel for the drop in consumption when r jumps to r^{high}.

One of the reasons for this large decline is the fact that individuals are myopic with respect to interest rate changes.[44] If the change in the interest rate was anticipated, the effect would be smaller as the discount rate employed by individuals would be a (time-varying) average of the low and the high interest rate. Nevertheless, consumption would still decline when the interest rate rises.

[44] While transitions in the employment state and the interest rate are both transitory by (2) and (4), interest rate changes are *perceived* as permanent shocks.

This figure also demonstrates where the endogenous scale dependence in our model comes from. As Gabaix et al. (2016), we allow for exogenous type dependence by working with different financial types.[45] This ex-ante heterogeneity then leads to endogenous scale dependence: High-financial-ability individuals experience a higher average growth rate of wealth as they are more often in the exploding regime, i.e. they are more often on the lower consumption path.

5.2 Targeting wealth distributions and measuring the fit

So far, we only talked about distributions of wealth for one individual that looks at some future point t in time. To obtain cross-sectional distributions from our model requires us to use a law of large numbers. When we assume that the number of individuals is sufficiently large within our cohort, the individual probability to own wealth below a certain threshold is the same as the share in the population (our 1979 cohort) of individuals holding this threshold or less. It also means that the individual probability p_i, introduced after (2), to be of a certain financial type equals the share p_i of individuals in the population to be of this financial type. We can therefore fit the aggregate wealth distributions to our individual densities as the latter also represent cross-sectional densities in our model.[46]

5.2.1 Targeting 2008

- The overall fit

As a starting point, we fit the model distribution to the wealth distribution in 2008. This requires three steps. First, starting from two initial subdensities $p^{\hat{z}}(\hat{a}, 0)$ for wealth in 1986, one for $\hat{z} = \hat{w}$ and one for $\hat{z} = \hat{b}$, we solve the Fokker-Planck equations in (23) employing optimal consumption paths $\hat{c}_r^{\hat{z}}(\hat{a}(t))$ shown in fig. 4.[47] For each of the n financial types, there is an infinity of realizations of possible interest rate paths. To make the numerical analysis simpler, we employ two interest rate paths for each financial type. These paths j are characterized by

[45] In contrast to Gabaix et al. (2016), our type dependence comes from an initial drawing of one's financial abilities. We do not model how individuals can switch types. In one of our robustness checks below we do find that data suggests that individuals switch types indeed.

[46] Stating laws of large numbers verbally is simpler than proving them. See He et al. (2017) for proposing a "nowhere equivalence" condition that allows to use Lebesgue integrals to model many economic agents.

[47] When we employ empirical densities from 1986 as initial conditions for our partial differential equation system, we need to make sure that our theoretical support in (24) is sufficiently large to cover the empirical range of observations. When we plot the empirical support for different years and compute the required initial support such that all observations are covered by the theoretical support, we obtain an initial support from $\hat{a}^{\text{nat}} = a^{\min}(1986) = -\frac{\hat{b}}{r-g}$ =-16,341US to $\hat{a}^{\max} = a^{\max}(1986) = a^{\max}_{\text{data}}(2008)/\Gamma(2008-1986)$ =1,020,400US$. As the highest empirical wealth observation for 1986 is 404,000 US $, we employ a density of zero for the range from this maximium empirical level to the required theoretical level. See app. C.2.3 for a plot of the empirical and the theoretical support.

an expected duration in the high regime which is consistent with the type's arrival rates λ_i^{high} and λ_i^{low} and differ in their initial interest rate level. We therefore solve the FPEs for the initial interest rate, say r_{low}, for as long as there is no jump in the interest rate. The resulting densities of wealth are then employed as initial densities for the next subperiod where the interest rate is r_{high} (see app. D.1).[48] We eventually obtain $2n$ wealth densities (n financial types times 2 possible initial interest rate levels) for 22 years later in 2008.[49]. The probability for an interest rate path is p_j, $j = 1, ..., 2n$. There is one subdensity for $\hat{a}_r^{\hat{w}}(22)$, one subdensity for $\hat{a}_r^{\hat{b}}(22)$ and the implied wealth density for $\hat{a}_r(22)$ from (22). Finally, for each interest rate path, we add the trend and obtain densities $g_j(a,t)$ from (25).

Second, given an exogenous number n of financial types, we determine population shares/ probabilities p_i to be of a certain financial type (via interest rate probabilities p_j) by maximizing our measure of fit,

$$F(t) = 1 - \frac{\int_{-\infty}^{\infty} \left| g^{\text{model}}(a,t) - g^{\text{data}}(a,t) \right| da}{2}. \tag{26}$$

The density predicted by the model,

$$g^{\text{model}}(a,t) = \Sigma_{j=1}^{2n} p_j g_j(a,t), \tag{27}$$

is the probability-p_j weighted sum of the $2n$ densities $g_j(a,t)$ from (25). The density obtained from the data is described by $g^{\text{data}}(a,t)$. Our measure of fit (which is related to the Kolmogorov–Smirnov statistic) starts from the absolute distance of model and data density as indicated by $|.|$. Imagine the densities do not have any overlap (like e.g. two uniform distributions one ending at x and the other one starting at $y > x$). We would then obtain $F(t) = 0$ as the integral over the densities would yield 2. The value of 0 would indicating no fit at all. By contrast, when the model density is identical to the data density, we would obtain $F(t) = 1$, indicating a perfect fit. With our cross-sectional interpretation, the probabilities $p_i^f = p_{2i-1} + p_{2i}$ to be of a certain financial type $i = 1, ..., n$ is equal to the share of individuals of that type. Fitting the wealth distribution in 2008 therefore means making a statement on how many financial types there are and how financial ability is distributed in our NLSY cohort.

Third, the optimal number n of financial types is chosen by computing measures of fit $F(2008)$ from (26) for $n \in \{2, ..., 130\}$ and selecting the number with the highest fit. This yields $n = 30$ and a plot of the fit as a function of n suggests that this is the unique maximum. Given two initial conditions each (starting with a high or with a low interest rate), this gives $2n = 60$ densities of wealth for 2008.

[48] As an alternative, we could simulate interest rate paths and compute the Monte Carlo average of the densities for each type. Given the good fit to be reported momentarily, we believe that simulations would not outperform our shortcut. It would be interesting to confirm this conjecture in future work.

[49] As our Fokker-Planck equations are linear, we solve them by employing the method of characteristics (see app. D.2). Consumption paths are obtained by a shooting algorithm. The matlab code is available at waelde.com/pub.

An illustration of the empirical fit is in fig. 5 for probabilities p_j that range from 0.3% to 8.0%.[50] The left figure shows the empirical density and 60 (unweighted) partial densities $g_j(a,t)$. Summing the (p_j-weighted) partial densities up as in (27), the right figure shows that the fit is almost perfect.[51] The figure also shows that the probability of owning wealth at the lower bound is zero. This is because we impose a natural borrowing limit, which is hit in finite time with a probability of zero.[52]

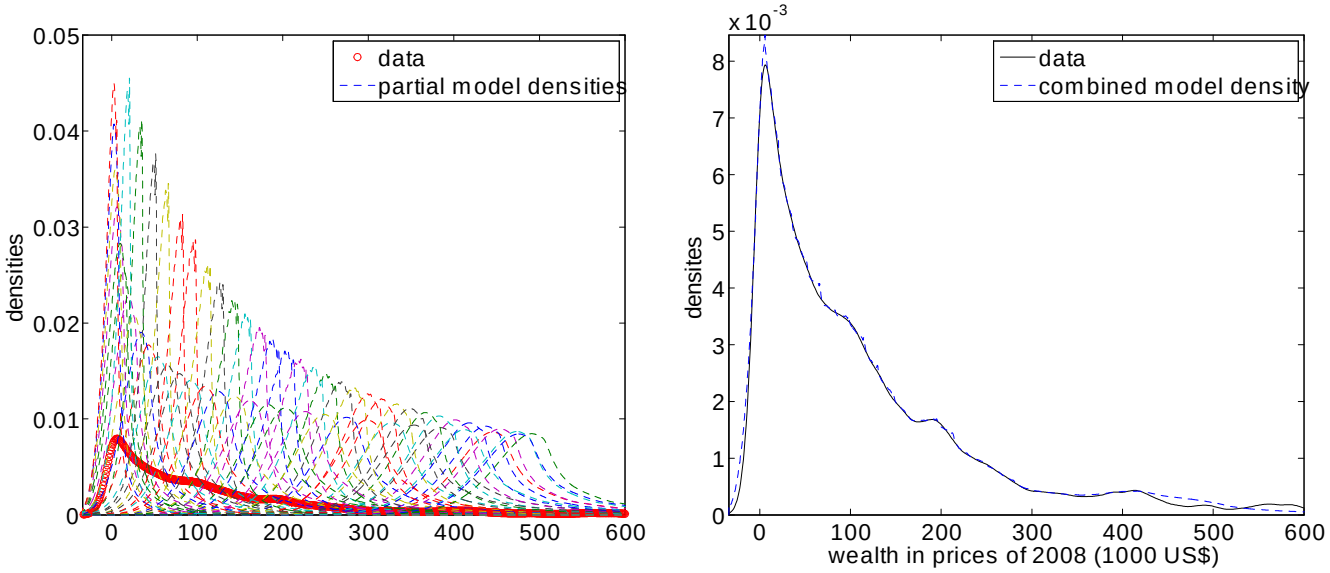

Figure 5 *The wealth distribution in 2008 in the data and in the model*

5.2.2 Measuring the fit for all years

When we choose p_j such that the fit in 2008 is maximized, the fit for waves between the initial distribution and 2008 is bound to be worse. The right figure in fig. 3 provides a visual impression of the fit between 1986 and 2008. As we would like to understand the fit also from a quantitative perspective, we compute our measure of fit $F(t)$ from (26) for all years. This yields the values displayed in the following table.

[50] The figure displays densities up to a wealth level of 600 (thousand US$) only. The support for 2008 is up to 2,123 (thousand US$), as described in app. C.2.3, but the visual impression for beyond 600 does not yield any insights.

[51] Increasing the number of financial types does not imply a fit of 100% as the range or width of each partial density is finite and does not become smaller as n increases. Hence, in contrast to the intuition behind an approximation of an integral by rectangles whose width reduces as the number of rectangles increases, here, we have an optimal number of financial types.

[52] The same argument is made by Achdou et al. (2020). Nuño and Moll (2018) impose a non-negative borrowing limit, which constitutes a tigher borrowing constraint. Since the probability to hit the lower bound of wealth is positive, it generates a spike at zero and mass to the right of zero of wealth distributions.

t	1986	1987	1988	1989	1990	1992	1994	1996	1998	2000	2004	2008
$F(t)$	100	73.5	63.0	60.6	61.4	66.3	72.1	77.2	81.9	84.4	87.4	96.1

Table 2 *The quantitative fit of the model according to (26) for target year 2008 in %*

The fit is perfect by construction for 1986 as we use the density from the data as initial distribution for the model. In terms of fig. 3, the empirical density in the left panel in 1986 is identical to the density in 1986 in the right panel. Between 1986 and 2008, the fit first falls and then rises. This is not a surprise as intermediate years were not targeted by the calibration. Finally, in 2008, the fit is close to perfect again. This can be visually checked again in fig. 3.

5.2.3 Targeting other years and targeting distributional dynamics

How would the fit improve if, starting in 1986, each year was targeted individually? The next table shows that the fit tends to increase over the years. The worst fit we obtain is 86% for 1987. The best fit now reaches 96.3% for 2004. The rise in the fit over time should be expected as the system, ceteris paribus, has more time to adjust to any given empirical wealth distribution.

t	1987	1988	1989	1990	1992	1994	1996	1998	2000	2004	2008
Fin. types (n)	8	40	40	52	30	34	28	22	30	30	30
$F(t)$	86.1	93.0	92.5	91.9	92.0	94.2	94.1	93.4	94.4	96.3	96.1

Table 3 *The quantitative fit (in %) of the model according to (26) where each year t is targeted individually*

We also targeted the dynamics of the wealth distribution by maximizing the average of $F(t)$ over all 11 waves from 1987 to 2008. The average $F(t)$ lies at 88.9%. We obtain a better average fit, as appears reasonable, as compared to the average over the fits in table 2 (which is 77.0%) when we target 2008. The individual fits range from 81.6% to 92.2% (see app. C.2.4 for a visual impression and the numbers).

5.3 Robustness checks

We undertook various robustness checks to understand how the fit would change when certain quantitative assumptions are adjusted. We also inquired into the relative role of capital income risk and labour income risk. We report the most relevant findings here.[53]

[53] In earlier calibrations, we divided the sample into 12 observationally distinguishable groups. In the absence of interest rate uncertainty, we were unable to match the upper tail of the wealth distribution. It is well-known that, unless one assumes a "superstar" or "awesome" state, this would be the case even if we allowed for more labour income states than just two. See Kaplan et al. (2018, footnote 35) for a similar argument.

5.3.1 Capital vs. labour income risk and types

What is the relative importance of capital and labour income risk from a quantitative perspective? Benhabib, Bisin and Zhu (2011, p. 133) write in their theoretical study "... that it is capital income risk (idiosyncratic risk on return on capital), and not labor income risk, that determines the heaviness of the tail of the stationary distribution given by the tail index: the higher is capital income risk, the more unequal is wealth". Benhabib, Bisin and Luo (2019) look at four channels, labour income risk, saving rates that differ across wealth levels, capital income risk and a rate of return of wealth that increases in wealth. They find that all the factors "have a fundamental role in generating the thick right tail of the wealth distribution" (p. 3).

In our analysis of the evolution of distributions over time, we focus on labour income risk and capital income risk. While our saving rates do change as a function of the wealth level of households, we can not switch this effect on and off as easily as Benhabib, Bisin and Luo (2019) can do in their two-period setup. We distinguish between three types of capital income risk (ex-ante, ex-post and financial types) and find *for our baseline model* that the interaction between capital income risk and labour income risk explains the evolution of wealth distributions over time. Neither labour income risk (which is well-understood), nor capital income risk on its own can explain fat right tails. With a *flexible interest rate distribution,* allowing for "awesome" or "superstar states" for capital income, we find that ex-ante capital income risk alone leads to a fit of 89.8% of the density in 2008. With all sources of capital income risk (ex-ante, ex-post and types), the fit increases to 96.7%. This fit is higher than the fit (of 96.1%) of our baseline model. Section 5.4 will make clear why we nevertheless consider the baseline model with the lower fit to be the most convincing calibration.

- The contribution of pure labour income risk

We employ our model to predict the effect of pure labour income risk. In this scenario, idiosyncratic labour income follows (4) but the process for interest rates (2) is switched off. We rather set the interest rate at 3.5% for all individuals or at 4.5% for all individuals.

As visible in the left panel of figure 6, an interest rate of 3.5% yields a wealth density in 2008 that is too far to the left. At the high interest rate 4.5%, given the non-stationary nature of the evolution of wealth, the density is too far to the right. The corresponding measures of fit are $F_{3.5}(2008) = 29.2\%$ and $F_{4.5}(2008) = 8.3\%$, correspondingly. The result that means are either too low or too high is not surprising. Yet, it is the lack of the spread that is crucial for the low fit. Hence, even when the constant interest rate were between 3.5% or 4.5%, the spread would always be too low.

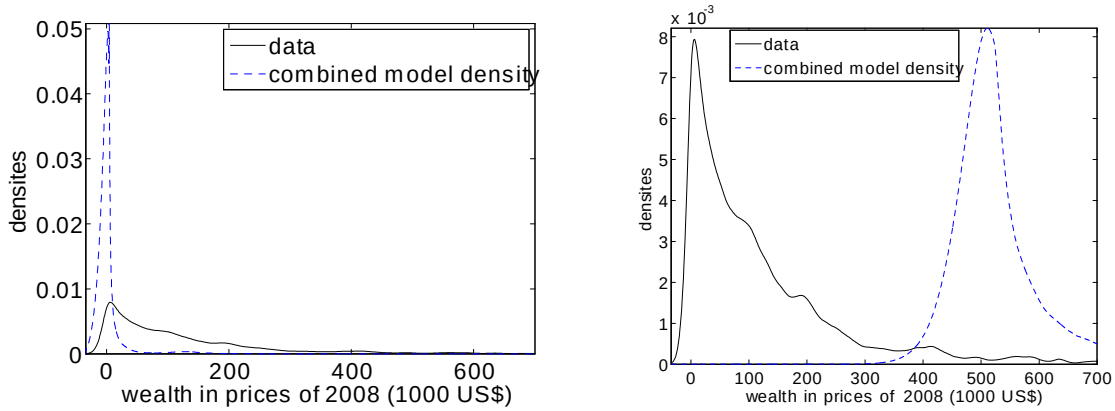

Figure 6 *The density of wealth for pure labour income risk at a constant interest rate of 3.5% (left figure) and at a constant interest rate of 4.5% (right figure)*

- The contribution of pure capital income risk

Let us now ask how the density of wealth looks like when we allow for capital income risk only. Let us remind ourselves about the sources of capital income risk. First, ex-ante heterogeneity results from individuals drawing an initial interest rate with $\text{Prob}\left(r\left(0\right)=r^{\text{low}}\right)=p_0$. Second, ex-post heterogeneity follows from interest rates fluctuating over time. Third, there is type dependence (Gabaix et al., 2016) as individuals belong to different financial types, i.e. they differ in their arrival rates λ_i^{low} and λ_i^{high} that govern the transition between the low and the high-interest-rate regime. Pure capital income risk means the absence of any wage distribution, neither ex-ante, nor ex-post.

We will therefore work with one wage

$$\tilde{w}\left(\tau\right)=u\left(\tau\right)b\left(\tau\right)+\left(1-u\left(\tau\right)\right)w\left(\tau\right) \tag{28}$$

which is a population-size weighted average of the wage $w\left(\tau\right)$ and unemployment benefits $b\left(\tau\right)$. The initial density for wealth in 1986 will be the empirical density for wealth (and not the usual sub-densities which do not apply in the presence of an average wage $\tilde{w}\left(\tau\right)$).

We display the density of wealth for pure capital income risk (at invariant labour income $\tilde{w}$) with (i) ex-ante heterogeneity and (ii) ex-ante and ex-post heterogeneity with two interest rate paths in app. C.3.1. When we add the third component of risky returns, financial types, we look at $2n$ interest rate paths and solve for densities of wealth after 22 years. We employ the 60 paths from the baseline model (left figure) but compute shares p_j optimally. The fit (shown in the right figure of fig. 7) is then 65.9%.

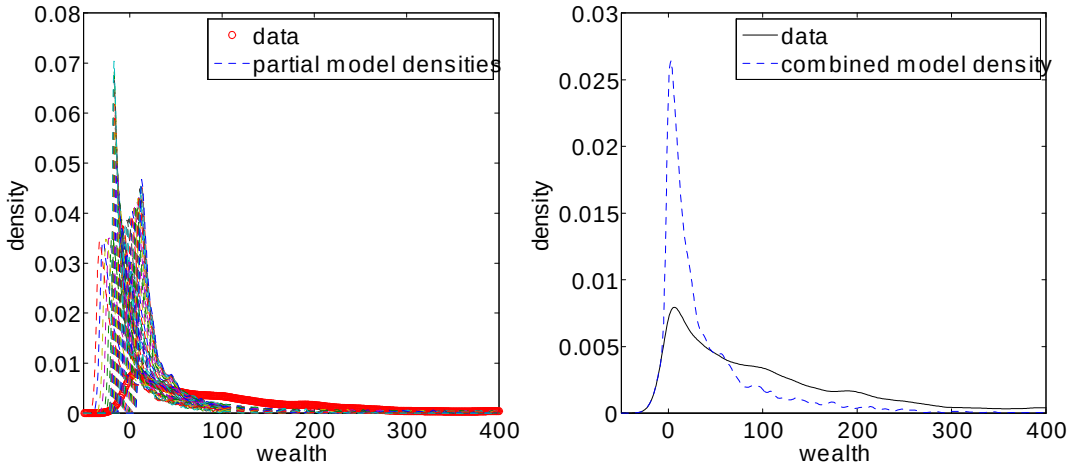

Figure 7 *The density of wealth for pure capital income risk at an invariant labour income* $\tilde{w}$ *with ex-ante and ex-post heterogeneity with* 60 *interest rate paths (left figure) and the overall density (right figure)*

Hence, just as pure labour income risk, capital income risk as presented in our baseline model is not enough to explain the dynamics of the distribution of wealth. It is the interaction of capital income risk and labour income risk that leads to a fit of above 90%.

- Extended ex-ante heterogeneity

We can also ask how the model would fit the wealth density in 2008 if we had an extended interest rate distribution that can take many values between 3.5% and 4.5%. This extension does not yield any significant improvement in the fit which stays at 64.7%. When we increase the upper bound to 8%, the fit increases to 82.4%. We obtain a further increase of the fit to 89.8% of the density in 2008 when we allow for 69 realizations between 3.5% and 15%. With these high "awesome" or "superstar realizations" (of close to 15%), ex-ante heterogeneity alone (i.e. an interest rate is drawn at the beginning of life and is kept constant thereafter) would almost be enough to reach the fit of the baseline model of 96.1%.

When we allow for all sources of capital income risk (ex-ante, ex-post and types), the fit increases to 96.7%. It therefore exceeds the fit of the baseline model with capital income risk (with two states) and labour income risk.[54]

[54]The broader conclusion from this analysis stresses how easily the effects of capital income risk can be overstated. When we remove labour income risk and allow for sufficient flexibility in the interest rate distribution, the model can still provide a very good (if not better) fit. Yet, too much of the variation in the wealth distributions would then be attributed to capital income risk and estimates might be biased.

- The role of types

Having understood the role of pure capital and pure labour income risk, we still need to understand the role of financial types in the baseline model. How does the fit change, when heterogeneity in financial types is removed in the full model with capital and labour income risk? Clearly, the fit depends on the arrival rates chosen for this specific financial type. In the best of all cases, the fit is $F(2008) = 67.8\%$. While this might sound like a good result, the qualitative fit, as the figure in app. C.3.2 shows, is not acceptable. Quantitatively, allowing for heterogeneity in types increases the fit up to the already reported 96.1%. Hence, allowing for types increases the fit in the baseline model by almost 30 percentage points or more than 40%. We conclude that allowing for type-heterogeneity is essential.

5.3.2 Wealth shares

When we target the *density* in 2008, the (non-targeted) wealth shares in the model in 2008 differ on average 3.9% from data wealth shares. For all waves, the average difference is at 7.6%. When we target the average over all years, the difference for 2008 increases to 5.7%. For all years, however, the average difference is 2.6% only.

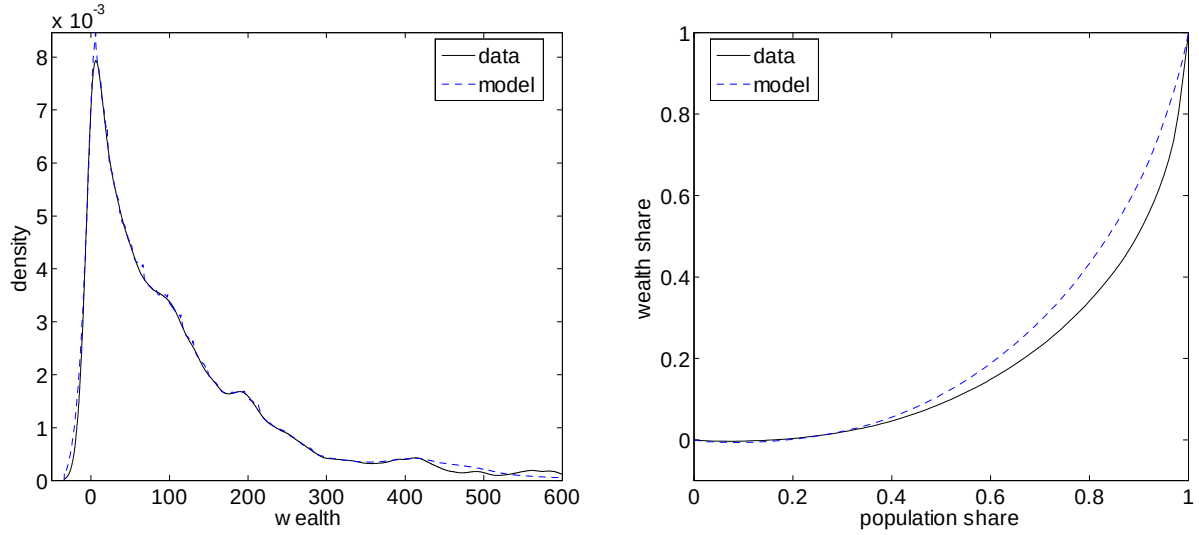

Figure 8 *Fit of the (targeted) density (identical to fig. 5) and the corresponding (non-targeted) Lorenz curve in 2008*

This figure provides a visual impression. The fit for the density is $F(2008) = 96.2\%$ as in fig. 5. To obtain a fit for the Lorenz curve, we measure the area A between the theoretical and the empirical Lorenz curve by $A \equiv \int_0^1 \left|\omega^{\text{model}}(x) - \omega^{\text{data}}(x)\right| dx$ where x is the population share and ω is the wealth share. The measure of fit is $0 < F^{\text{Lorenz}}(t) = 1 - 2A < 1$. It equals 1 when $A = 0$ (the two Lorenz curves coincide) and equals 0 when $A = 1/2$.[55] The fit for the

[55] The measure is not the difference between the Gini-coefficient in the model and in the data. Obviously, for one Gini coefficients there is an infinity of different Lorenz curves.

Lorenz curve is then $F^{\text{Lorenz}}(2008) = 92.0\%$.

Figure 9 shows the fit, when we target *wealth shares* in 2008. We obtain the model Lorenz curve by starting from the 60 interest rate paths in our baseline model. The densities of these paths are visible in fig. 5. Then we optimally choose probabilities $p_j^{\text{Lorenz2008}}$ such that the area A between the theoretical and the empirical Lorenz curve is minimized. According to our distance measure, the curves coincide by $F^{\text{Lorenz}}(2008) = 99.5\%$. The corresponding density in 2008, visible to the right in fig. 9, however, shows that the density fit is not very convincing when wealth shares are targeted. Employing the measure from (26), we find a value of $F(2008) = 74.5\%$.

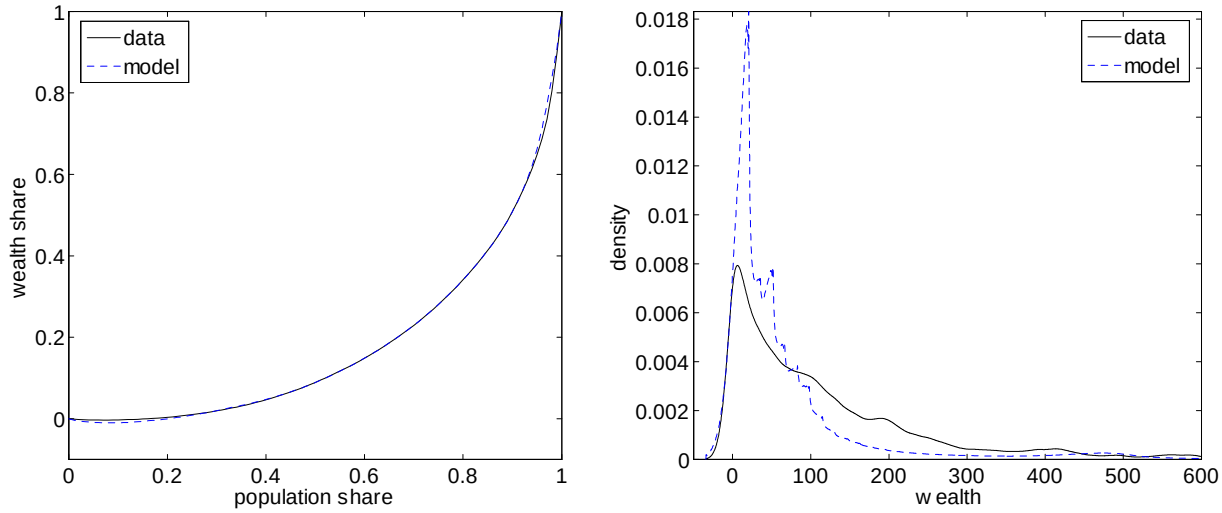

Figure 9 *Fit of the (targeted) Lorenz curve and the corresponding fit of the (non-targeted) density in 2008*

This finding shows the strong trade-off between fitting densities and Lorenz curves. It also shows how useful it is to introduce Gabaix et al. (2016) types also for quantitatively fitting Lorenz curves. When one is interested in a good fit of both the density and wealth (or other) shares and only one object is targeted, the density as a target seems to yield the better overall fit.

5.3.3 The effect of the high interest rate and of risk aversion

One might inquire into the effect of a broader range of the idiosyncratic interest rate. We therefore targeted 2008 under a high interest rate of 8% instead of 4.5%. All other parameters were left unchanged. This implies that a_b^* moves to the left (66, 273US$ instead of 930, 132 US$ as visible in fig. 4) and the fit increases slightly to $F(2008) = 97.3\%$. (For more details, see app. C.3.4.) As with a rate of 4.5%, the unemployed accumulate wealth beyond a_b^*. As this range is now much larger, the right tail becomes fatter. Overall, however, our general findings are confirmed.

As discussed after fig. 4, the drop in consumption in the high-interest-rate regime is due to the drop in the present value of labour income. We nevertheless inquire into the effect of risk aversion on our findings. When we set σ equal to 0.8, the fit $F(2008)$ drops to 90.3%. This is still a reasonable value and there is enough probability mass in the right tail as $\sigma = 0.8$ still satisfies $r_{\text{low}} < \rho + \sigma g < r_{\text{high}}$. There is a low-interest-rate regime and the high-interest-rate regime is actually an exploding regime (see app. C.3.5). For risk aversion equal to 1.2, the fit falls dramatically to $F(2008) = 44.7\%$ (even though the average fit over all years is still at 69.8%). This follows from the fact that for $\sigma = 1.2$, the interest rates satisfy $r_{\text{low}} < r_{\text{high}} < \rho + \sigma g$. Hence, there is no longer any exploding regime, both regimes are low-interest-rate regime, all wealth is below a_w^* from fig. 1 and there is a very thin right tail (see again app. C.3.5 for a visual impression).

5.3.4 Thickness of the right tail

We also study how well our model replicates thick right tails that we observe in empirical wealth distributions. The literature typically estimates the Pareto coefficient of this tail and treats this coefficient as a measure of thickness. The lower the coefficient, the fatter the tail. When this approach is applied to city size, firm size or income, the typical log-log plots (see e.g. Gabaix, 2009) provide a straight line. One well-known issue of this approach (discussed e.g. by Atkinson, 2017, in the context of income distributions) consists in fixing where actually the right tail starts. Another, less discussed issue revolves around possible right-censoring. This has been reported by Aban et al. (2006) for applications in finance, hydrology and atmospheric science. This feature can be suspected in the data when the log-log plot displays a strong concavity after some threshold. Aban et al. also propose a solution to this issue by estimating a truncated Pareto distribution.

We indeed find typical features of right-censorship in the NLSY wealth distributions (see app. C.3.7). After some threshold that differs across survey waves, the straight line curves strongly downwards.[56] For the wealth distribution in 2008, this concavity starts at the level of about 1.34 million US Dollar. One possible reason for this lies in the questionnaire nature of the NLSY dataset. Individuals with very high wealth levels might just report lower levels.[57] Given this strong concavity, we calibrate the parameters of a right-censored Pareto density as presented in Aban et al. (2006, eq. (3)).

Employing this (right-censored) Pareto density for the right tail, we provide measures of its thickness, the so-called Pareto coefficients. The economics literature reports Pareto coefficients for wealth of around 1.5 (Gabaix, 2009, p. 275) and 2 (Cao and Luo, 2017, table 2), usually

[56] This is what Atkinson (2017, fig. 1) calls the 'Baronial shape'.

[57] Looking at the plot in the online appendix H of Cao and Luo (2017) suggests that right censoring might also be present in the SCF dataset.

not taking right-censoring into account.[58] For comparison purposes, we report the slope of the straight line that follows from our estimation of the truncated Pareto distribution. The slope is measured from the 80th percentile to the 90th percentile. Beyond the 90th percentile, the effect of the truncation sets in strongly.[59] Our empirical measures of thickness (i.e. the Pareto coefficient) are reported in app. C.3.7, an illustration is in figure 10.

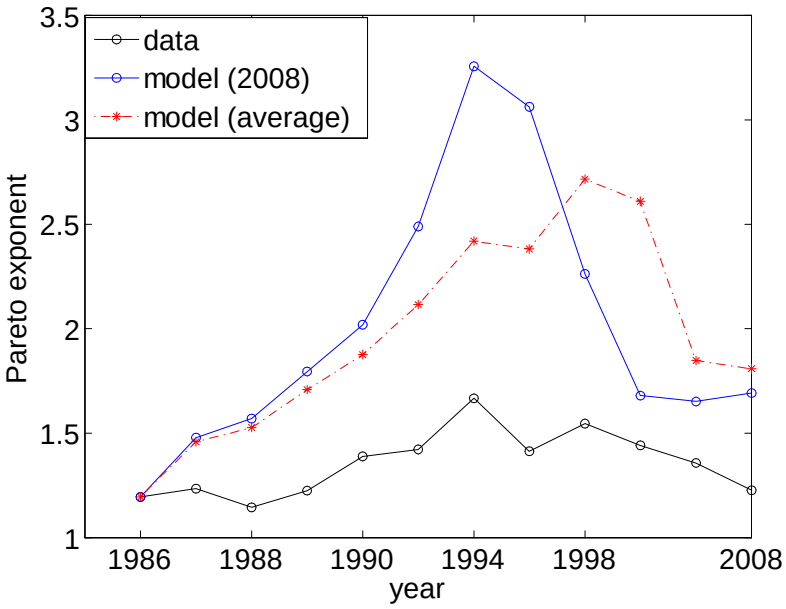

Figure 10 *Pareto coefficients (non-targeted) in data and model for two different calibrations*

The empirical and model Pareto coefficients are by construction identical in 1986. The measures are very close in 2008, both for the calibration where we target the density in 2008 and for the calibration where we target the average over all years. As expected, the Pareto-coefficient is closer in 2008 for the calibration where we target the density in 2008. Targeting the average density over all years implies better average Pareto coefficients, however: The model coefficient exceeds the data coefficient on average by 46.5% when we target 2008. When we target the average fit of all densities, the average fit of Pareto coefficients is higher (with an average excess of 44.1% only). More generally, if we targeted wealth shares (and not densities), the Pareto coefficients would fit even better. As our objective consisted in targeting densities, we consider this fit in Pareto coefficients as very good.

[58] When we apply the short-cut based on wealth shares employed by Cao and Luo (2017, footnote 1 and section 6.1) on wealth data from Saez and Zucman (2016), we find Pareto coefficients (for the range from the 90th to the 99th percentile for the period from 2000 to 2012) that lie between 1.36 and 1.50.

[59] There is no obvious rule - as argued by Atkinson (2017) - where the right tail starts. We choose the 80th percentile as wealth seems to be almost perfectly Pareto distributed in *all* waves as of the 80th (and up to the 90th) percentile.

5.3.5 Is financial ability time-invariant?

Financial ability i of one individual is captured by a pair of arrival rates λ_i^{low} and λ_i^{high}. These arrival rates describe how quickly on average an individual moves from low to high returns (and back). The transition rates capture the deeper idea that individuals are born or enter their economically active life with certain skills which, given some economic environment, imply this pair of arrival rates. When we look at a period of 22 years with big changes on financial markets over this period (think of the dot-com bubble in the late 1990s or the direct access for private and small investors via the internet to almost all asset types), it would be hard to argue that financial ability i is invariant over these 22 years. We therefore also inquired into potential breaks in the distribution of financial ability.

Our starting point is the fit $F(2008)$ for 2008 with 30 financial types of 96.1% in table 3. When we employ the quantitative weights p_i of these 30 financial types (see app. C.3.8), we find that individuals spent 36.3% of their time in the high-interest-rate regime. Hence, on average, individuals experience 8 years (36.3% out of 22 years) of high interest rates. When we take the same number of financial types and fit 1998, we find that individuals spent 47.1% of their time in the high-interest-rate regime.

Changes in average returns over time can have many reasons. Individual learning or simpler access to financial markets over time for this cohort are obviously not strong enough as periods of high returns fall after 1998. We conclude that the positive effects of learning or lower transaction costs is overcompensated by falling average idiosyncratic returns after 1998.

5.4 The distribution of idiosyncratic interest rates

We have presented various quantitative versions of our model. The most relevant ones yield a fit of the empirical density of wealth in 2008 of around 90%. We can "test" these calibrations by inquiring whether the idiosyncratic interest rate distributions in the model have properties that are broadly consistent with empirical idiosyncratic interest rate distributions. We focus on the baseline model, the model with ex-ante capital risk only and on the model with three sources of capital income risk (ex-ante, ex-post and types).

The empirical evidence is summarized in table 4.[60] The means range from slightly negative values to values up to 14%. The (unweighted) average mean from this table is 5.7%. The standard deviations all lie above 3% with the highest estimate above 27%. The average standard deviation is 12.1%.

Turning to capital income risk in our baseline model, given that $r(t) \in \{3.5\%,\ 4.5\%\}$ and type i specific arrival rates λ_i^{low} and λ_i^{high}, we can compute the probabilities $\pi_i(\tau) \equiv \text{Prob}(r_i(\tau) = r_{\text{high}})$

[60] The table either displays all assets reported in these studies or representative ones. Our general conclusion drawn below does not depend on this selection.

that a financial type i has a high interest rate at a point in time τ. Using the population shares p_i, we can predict the unconditional probability for an investor that the interest rate is high, $\pi(\tau) \equiv \text{Prob}(r(\tau) = r_{\text{high}})$. For any fit, we can therefore compute time paths of moments and compare them to empirical moments. For our target year 2008, we obtain a annual mean return of 4.4% with a standard deviation of 0.44%. Standard deviations for our robustness checks are of the same order of magnitude. The highest one is generated when the high interest rate is at 8%. Even then, the standard deviation is only 1.34%.[61]

Asset	Mean	St.dev.	Country	Source
T-bills	−0.38%	4.35%	US	Flavin and Yamashita (2002)
Bonds	0.60%	8.40%		PSID: 1968 to 1992
Stocks	8.24%	24.15%		S&P 500: 1926 to 1992
Mortgage	0.00%	3.36%		
House	6.59%	14.24%		
Wealth (1)	7.92%	27.14%	US	Cao and Luo (2017)
Wealth (2)	5.94%	11.27%		PSID: 1984, 1989 and 1994
Private equity	13.1%	6.90%	US	Moskowitz and
Public equity	14.0%	17.00%		Vissing-Jorgensen (2002)
Financial wealth	4.19%	14.35%	Norway	Fagereng et al. (2018, tab. 3)
Housing	4.59%	6.09%		Administrative tax data: 1993 to 2013
Net worth (3)	3.66%	7.46%		

(1) with capital gains, (2) without capital gains, (3) after tax

Table 4 *Empirical idiosyncratic interest rate distributions*

Is our low standard deviation a quantitatively interesting finding or an artefact of our assumed structure where the idiosyncratic interest rate can take only two values, r^{low} and r^{high}? When we replace this discrete distribution by a continuous uniform distribution, the standard deviation is given by $\sigma^{\text{uniform}} = \left(r^{\text{high}} - r^{\text{low}}\right)/\sqrt{12}$. With our values of 3.5% and 4.5%, the standard deviation amounts to 0.29%. Hence, our findings do not seem to be driven by the discrete and simple distribution of the interest rate.

When we turn to the ex-ante capital risk calibration, the best fit is obtained for $2.5\% \leq r_i(0) \leq 8\%$ with $n = 22$ equidistant realizations of $r(0)$. The probabilities p_i to draw an $r_i(0)$ imply a mean of 7.99%. This exceeds means in empirical interest rate distributions.

[61] See also Bach et al. (2015) who use administrative data of Swedish residents. They find that the heterogeneity in returns is attributed to not only the allocation of wealth but also the level of wealth. They document that returns on financial wealth are on average 4% higher per year for households in the top 1% compared to the median household.

When we take the same number of paths as in our baseline model, allow for all three sources of capital income risk as in fig. 7 with an invariant labour income $\tilde{w}$ from (28) but increase the upper bound of the interest rate to 15%, i.e. $r(t) \in \{3.5\%, 15\%\}$, the mean of the interest rate distribution in 2008 is 10.1%. Again, this is larger than the mean in empirical interest rate distributions.[62]

Summarizing, we can construct interest rate distributions where capital income risk alone, without any labour income risk, can generate extremely good fits for the dynamics of the distribution of wealth. These interest rate distributions include "awesome" or "superstar states", however, and are therefore empirically not convincing. When we turn to distributions of interest rates that have reasonable average idiosyncratic returns and combine them with empirically convincing labour income risk, we obtain a level of fit (96.1%) which is highly satisfactory. We do stress, however, that this baseline calibration seems to "overexplain" wealth inequality as the standard deviation of the interest rate distribution in the baseline model is considerably lower than in the data.[63]

6 Conclusion

This paper began by describing an optimal saving model for an individual facing idiosyncratic labour income shocks and idiosyncratic capital income shocks. Labour income grows over time but interest rates are stationary. Interest rates fluctuate between a value that implies a stationary wealth distribution, as in standard precautionary savings models, and a value that implies non-stationary wealth distributions (a so-called "exploding regime"). In addition to the two sources of ex-post heterogeneity, labour and capital income risk, we allow for ex-ante heterogeneity in financial abilities of individuals.

We solve the FPEs to describe the evolution of wealth for one individual and thereby also for a cross-section of individuals of identical individuals. When we aggregate over different financial types, we obtain a distribution of wealth that evolves over time and that can be used to understand the wealth distributions of the NLSY 79 cohort. Our agents form rational expectations (subject to our numerical caveat from footnote 29) and no approximation techniques

[62] Future work could go beyond comparing means and standard deviations in models and data. Autocorrelation and other properties of stochastic processes should be modelled and taken into account as well. Empirical analyses seem to suggest that individual fixed effects for idiosyncratic interest rates could be augmented by AR(1) or, more convincing, MA(2) processes for the error term. If, in addition, regime switching processes could be estimated, more flexible distributional assumptions than in AR or MA processes could be allowed for. This would bring theory (generalizing our 2-state process for the interest rate to n states) and empirical analyses closer together. We are grateful to Luigi Pistaferri for discussions of their findings in Fagereng et al. (2018).

[63] A next step in the analysis of this conjecture would work with a theoretical structure that is rich enough to allow households to invest in as many assets as reported by the studies sumarized in tab. 4.

are required to study the evolution of distributions over time.

We quantify our model by employing parameter values that imply, for example, wage levels and wage growth that are consistent with empirical values from the NLSY. The initial densities of wealth for our model are taken from the 1986 wave of the NLSY. By computing the share of individuals that have a certain financial ability, our model density for 2008 overlaps with the empirical density by more than 96%. For intermediate years, the fit can fall down to 60.6%. When we maximize the fit for all 12 waves with wealth information, the average fit is 88.9%.

The fat right tail of wealth distributions can be understood by a quantitative version of qualitative phase diagrams for the two interest rate regimes. Optimal consumption level drops strongly in the exploding regime compared to the low-interest-rate regime. This drop yields fast wealth accumulation (at least for employed workers) and, therefore, moves sufficiently many individuals into the right tail of the wealth distribution.

Computing the shares of financial abilities yields a prediction of capital income risk. When we check the empirical plausibility of the quantitative interest rate distribution in our model, we find that the standard deviation for interest rates needed in capital income risk models to generate plausible wealth distributions with fat right tails is much lower than what is observed empirically. The capital income risk approach to understanding wealth distributions, therefore, also seems to be promising from a quantitative perspective.

When we compare our baseline model to a model with pure capital income risk, we find that such a model (that abstracts from any labour income risk) can generate an even higher fit for the evolution of wealth. Yet, this high fit comes at a cost of having to allow for "superstar states" in the interest rate distribution; that is, for returns that are empirically not convincing.

Future work should allow for continuous wage and interest rate distributions. This would generalize our approach and also allow agents to form expectations about uncertain interest rates without additional numerical complexities. Studying the dynamics of wealth distribution in general equilibrium would be another interesting project. While this has been done in the past, standard formation of expectations still needs to be taken into account in numerical methods. We are confident that an approach based on FPEs can help in reaching this goal.

References

Aban, I. B., M. M. Meerschaert, and A. K. Panorska (2006): "Parameter Estimation for the Truncated Pareto Distribution," Journal of the American Statistical Association, pp. 270 – 277.

Achdou, Y., F. Buera, J. Lasry, P. Lions, and B. Moll (2014): "Partial differential equation models in macroeconomics," Philosophical Transactions of the Royal Society A 372: 20130397,

pp. 1–19.

Achdou, Y., J. Han, J. Lasry, P. Lions, and B. Moll (2020): "Income and Wealth Distribution in Macroeconomics: A Continuous-Time Approach," mimeo Princeton University.

Ahn, S., G. Kaplan, B. Moll, T. Winberry, and C. Wolf (2017): "When Inequality Matters for Macro and Macro Matters for Inequality," CESifo Working Papers, CESifo Working Paper No. 6581.

Angeletos, G. M. (2007): "Uninsured idiosyncratic investment risk and aggregate saving," Review of Economic Dynamics, 10(1), 1–30.

Angeletos, G. M., and L. E. Calvet (2005): "Incomplete-market dynamics in a neoclassical production economy," Journal of Mathematical Economics, 41(4-5), 407–438.

——— (2006): "Idiosyncratic production risk, growth and the business cycle," Journal of Monetary Economics, 53(6), 1095–1115.

Aoki, S., and M. Nirei (2017): "Zipf's Law, Pareto's Law, and the Evolution of Top Incomes in the United States," American Economic Journal: Macroeconomics 2017, 9(3), 36–71.

Atkinson, A. B. (2017): "Pareto and the Upper Tail of the Income Distribution in the UK: 1799 to the Present," Economica, 84(334), 129–156.

Bach, L., L. E. Calvet, and P. Sodini (2015): "Rich Pickings? Risk, Return, and Skill in the Portfolios of the Wealthy," Working paper.

Bayer, C., A. D. Rendall, and K. Wälde (2019): "The Invariant Distribution of Wealth and Employment Status in a Small Open Economy with Precautionary Savings," Journal of Mathematical Economics, 85, 17–37.

Bayer, C., and K. Wälde (2010a): "Matching and Saving in Continuous Time: Proofs," CESifo Working Paper 3026-A.

——— (2010b): "Matching and Saving in Continuous Time: Theory," CESifo Working Paper 3026.

——— (2015): "The Dynamics of Distributions in Continuous-Time Stochastic Models," mimeo - available at www.waelde.com/pub.

Benhabib, J., and A. Bisin (2018): "Skewed Wealth Distributions: Theory and Empirics," Journal of Economic Literature, 56, 1261–1291.

Benhabib, J., A. Bisin, and M. Luo (2017): "Earnings Inequality and Other Determinants of Wealth Inequality," American Economic Review: Papers & Proceedings, 107(5), 593–597.

——— (2019): "Wealth Distribution and social mobility in the US: A quantitative approach," American Economic Review, 109, 5, 1623–47.

Benhabib, J., A. Bisin, and S. Zhu (2011): "The Distribution of Wealth and Fiscal Policy in Economies with Finitely Lived Agents," Econometrica, 79(1), 123–157.

——— (2015): "The wealth distribution in Bewley economies with capital income risk," Journal of Economic Theory, 159, 489–515.

——— (2016): "The Distribution of Wealth in The Blanchard-Yaari Model," Macroeconomic Dynamics, 20(2), 466–481.

Birkner, M., and K. Wälde (2014): "The Dynamics of Distributions - Applying Stochastic Processes in the Social Sciences," Department of Mathematics and Department of Economics, Johannes Gutenberg University Mainz.

Blundell, R., M. Graber, and M. Mogstad (2015): "Labor income dynamics and the insurance from taxes, transfers, and the family," Journal of Public Economics, 127, 58 73.

Cao, D., and W. Luo (2017): "Persistent Heterogeneous Returns and Top End Wealth Inequality," Review of Economic Dynamics, 26, 301–326.

Castaneda, A., J. Diaz-Gimenez, and J.-V. Rios-Rull (2003): "Accounting for the U.S. Earnings and Wealth Inequality," Journal of Political Economy, 111, 818–857.

Diamond, P. A. (1982): "Aggregate Demand Management in Search Equilibrium," Journal of Political Economy, 90, 881–94.

Evans, M., N. Hastings, and B. Peacock (2000): Statistical Distributions, 3rd edition. Wiley.

Fagereng, A., L. Guiso, D. Malacrino, and L. Pistaferri (2018): "Heterogeneity and Persistence in Returns to Wealth," CESifo Working Paper No. 7107.

Flavin, M., and T. Yamashita (2002): "Owner-Occupied Housing and the Composition of the Household Portfolio," American Economic Review, 92(1), 345–362.

Gabaix, X. (2009): "Power Laws in Economics and Finance," Annual Review of Economics, 1, 255–293.

Gabaix, X., J.-M. Lasry, P.-L. Lions, B. Moll, and Z. Qu (2016): "The Dynamics of Inequality," Econometrica, 84(6), 2071–2111.

Guvenen, F., F. Karahan, S. Ozkan, and J. Song (2019): "What Do Data on Millions of U.S. Workers Reveal about Life-Cycle Earnings Dynamics?," Federal Reserve Bank of New York Staff Report, 710.

Hall, R. E., and P. R. Milgrom (2008): "The Limited Influence of Unemployment on the Wage Bargain," American Economic Review, June 2008.

He, W., X. Sun, and Y. Sun (2017): "Modeling infinitely many agents," Theoretical Economics, 12(2), 771–815.

Hubmer, J., P. Krusell, and A. Smith (2019): "The Historical Evolution of the Wealth Distribution: A Quantitative-Theoretic Investigation," mimeo extending NBER Working Paper 23011.

Johnson, N., S. Kotz, and N. Balakrishnan (1994): "Continuous Distributions (General)," in Continuous Univariate Distributions Vol. 1, ed. by N. Johnson, S. Kotz, and N. Balakrishnan, pp. 1–79. Wiley Publications.

Jones, C. (2015): "Pareto and Piketty: The Macroeconomics of Top Income and Wealth Inequality," Journal of Economic Perspectives, 29(1), 29–46.

Jones, C., and J. Kim (2017): "A SchumpeterianModel of Top Income Inequality," Journal of Political Economy, forthcoming.

Kaplan, G., B. Moll, and G. L. Violante (2018): "Monetary Policy According to HANK," American Economic Review, 108, 697–743.

Kasa, K., and X. Lei (2018): "Risk, uncertainty, and the dynamics of inequality," Journal of Monetary Economics, 94, 60–78.

Kaymak, B., and M. Poschke (2016): "The evolution of wealth inequality over half a century: The role of taxes, transfers and technology," Journal of Monetary Economics, 77, 1–25.

Krusell, P., and A. A. Smith, Jr. (1998): "Income and Wealth Heterogeneity in the Macroeconomy," Journal of Political Economy, 106, 867–896.

Launov, A., and K. Wälde (2013): "Estimating Incentive and Welfare Effects of Non-Stationary Unemployment Benefits," International Economic Review, 54, 1159–1198.

Lise, J. (2013): "On-the-Job Search and Precautionary Savings," Review of Economic Studies, 80, 1086–1113.

Mankiw, N. (2015): "Yes, r $>$ g. So What?," American Economic Review: Papers & Proceedings, 105(5), 43–47.

Mattheij, R. M. M., S. W. Rienstra, and J. H. M. ten Thije Boonkkamp (2005): Partial Differential Equations: Modeling, Analysis, Computation. Society for Industrial & Applied Mathematics.

Merton, R. C. (1971): "Optimum Consumption and Portfolio Rules in a Continuous-Time Model," Journal of Economic Theory, 3(4), 373–414.

Mortensen, D. T. (1982): "Property Rights and Efficiency in Mating, Racing, and Related Games," American Economic Review, 72, 968–79.

Moskowitz, T. J., and A. Vissing-Jorgensen (2002): "The Returns to Entrepreneurial Investment: A Private Equity Premium Puzzle?," American Economic Review, 92, 745–778.

Nagel, T. (2013): "Poverty and Labour Market Policies in a Stochastic and Dynamic World," Dissertation, Johannes Gutenberg University Mainz. Available at www.waelde.com/pub.

Nardi, M. D., and G. Fella (2017): "Saving and Wealth Inequality," Review of Economic Dynamics, 26, 280–300.

Nirei, M., and S. Aoki (2016): "Pareto distribution of income in neoclassical growth models," Review of Economic Dynamics, 20, 25–42.

Nuño, G., and B. Moll (2018): "Social optima in economies with heterogeneous agents," Review of Economic Dynamics, 28, 150–180.

Parra-Alvarez, J., O. Posch, and M.-C. Wang (2017): "Identification and estimation of heterogeneous agent models: A likelihood approach," CREATES Research Paper, (2017-35), 1–41.

Piketty, T. (2015a): "About Capital in the Twenty-First Century," American Economic Review: Papers & Proceedings, 105(5), 48–53.

Piketty, T. (2015b): "Putting Distribution Back at the Center of Economics: Reflections on Capital in the Twenty-First Century," Journal of Economic Perspectives, 29(1), 67–88.

Piketty, T., and G. Zucman (2015): Wealth and Inheritance in the Long Run,pp. 1304–1366. Handbook of Income Distribution, Atkinson, A. and Bourgignon, F. (eds.), Volume 2B, 2015 Elsevier B.V.

Pissarides, C. A. (1985): "Short-run Equilibrium Dynamics of Unemployment Vacancies, and Real Wages," American Economic Review, 75, 676–90.

Piyapromdee, S. (2018): "Residual Wage Dispersion with Efficiency Wages," International Economic Review, 59(3), 1315–1343.

Pröhl, E. (2017): "Approximating Equilibria With Ex-Post Heterogeneity and Aggregate Risk," Swiss Finance Institute Research Paper No. 17-63, available at http://dx.doi.org/10.2139/ssrn.2620937.

Saez, E., and G. Zucman (2016): "Wealth Inequality in the United States since 1913: Evidence from Capitalized Income Tax Data," Quarterly Journal of Economics, pp. 519 – 578.

Sennewald, K. (2007): "Controlled Stochastic Differential Equations under Poisson Uncertainty and with Unbounded Utility," Journal of Economic Dynamics and Control, 31, 1106–1131.

Shephard, A. (2017): "Equilibrium Search and Tax Credit Reform," International Economic Review, 58(4), 1047–1088.

Strikwerda, J. C. (2004): Finite Difference Schemes and Partial Differential Equations. SIAM - Society for Industrial and Applied Mathematics, Philadelphia, 2nd edn.

Wackerly, D., W. Mendenhall, and R. Scheaffer (2008): Mathematical Statistics with Applications, 7th ed. Thomson Brooks/Cole.

Wälde, K. (2012): Applied Intertemporal Optimization. Know Thyself - Academic Publishers, available at www.waelde.com/KTAP.