Advanced Macroeconomic Theory 1 Part 2

Prof. Dr. Klaus Wälde
Hoang Van Khieu

Mock exam

Notes

- Please fill in your details on this front page below and on all pages of your answer sheets as well.
- Solution sheets are handed out before the exam. Only answers on the solution sheets will be scored.
- Please indicate *clearly* which question and sub-question you answer to avoid confusion.
- Please keep the solution sheet face down on the table until instructed otherwise.
- Please make sure your mobile is switched off, and your personal belongings are not by your side.
- This is a closed-book exam. *No material* is allowed. You are not allowed to communicate with each other, nor with anyone else outside of this room.
- Please provide your student card and your ID, which will be checked during the exam.
- The examination lasts exactly *60 minutes*. Once the examination time is over, please put down your pen and stop writing immediately.
- Upon completing the exam, please put your exam sheets together with your answer sheets, and wait for them to be collected.

This is a collection of questions which were _not_ jointly asked in one exam. These questions were asked in different exams. They are both of the "creative type", i.e. they require a transfer of knowledge from methods learned to methods applied to new questions.

As they were not jointly asked in one exam, this collection is not representative of typical exams. It has a pro-creative bias. We hope the collection nevertheless helps for preparing for exams.

Question 1 - Green growth: non-renewable resources

Consider a central planner maximising the objective function

$$U(0) = \int_0^\infty e^{-\rho t} \frac{C(t)^{1-\sigma} - 1}{1-\sigma} dt, \tag{1}$$

where $1/\sigma > 0$ is the intertemporal elasticity of substitution, $\rho > 0$ is the time preference rate, and aggregate consumption $C(t) \geq 0$ is given by

$$C(t) = A(t)R(t), \tag{2}$$

where $A(t) = A_0 e^{gt} \geq 0$ is productivity, with growth rate $g > 0$ and initial level $A_0 > 0$, and $R(t) \geq 0$ is a flow of exhaustible resources (e.g. oil, gas, coal). In other words, the economy uses exhaustible resources to produce the consumption good at current productivity, which is then consumed by the representative individual.

The flow of the exhaustible resource $R(t)$ reduces its stock $S(t)$. The stock evolves over time according to the differential equation

$$\dot{S}(t) = -R(t) \tag{3}$$

where $S(t) \geq 0$ is required.

1. Show that the KRR for $R(t)$ is given by

$$\frac{\dot{R}(t)}{R(t)} = \frac{(1-\sigma)g - \rho}{\sigma}$$

   and give an interpretation to the expression based on the sign of the KRR.

2. Plot the phase diagram of the model using the differential equations for $R(t)$ and $S(t)$.

3. Under which conditions does consumption $C(t)$ grow over time?

4. As we would like to qualitatively match real world stylised facts, find a parameter condition under which (i) consumption grows and (ii) resources fall over time.

Assume that the solution to the KRR for $R(t)$ above is given by $R(t) = R_0 e^{g_R t}$, where R_0 is (exogenously given) initial resource use and g_R is the growth rate of $R(t)$. Further, assume that $g_R < 0$.

5. Use the solution for $R(t)$ with the law of motion $\dot{S}(t)$ and compute the closed-form solution of $S(t)$ as a function of R_0, S_0, and g_R. Give an interpretation to the solution for any t and for the case when t goes to infinity.
   (*Note: the differential equation $\dot{x}(t) = ax(t) - b(t)$ admits a unique backward solution $x(t) = x_0 e^{at} - e^{at} \int_0^t e^{-as} b(s) ds$.*)

6. Plot $S(t)$ as a function of time t. Assuming that we eventually run down the stock $S(t)$, compute the point in time $t > 0$ when we have $S(t) = 0$. What could be done to push this point in time t further into the future?

Question 2 - Epidemics and decisions

Consider an individual who can be in three states: healthy, sick and recovered. In more traditional epidemiological terms, the states would be called "suzeptible", "infectious" and "removed" - which gives the acronym SIR. For simplicity, we denote the states by 1, 2 and 3, respectively. The individual is initially in state 1, at some point moves into state 2 and then ends up in state 3, as illustrated in Figure 2. Technically, state 1 is an emitting state (no individual ever enters this state) and state 3 is an absorbing state (no individual ever leaves this state).

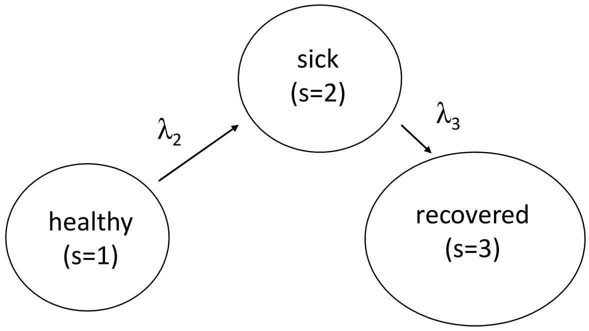


Figure 1 *The SIR model*

The rate for the individual to get infected is given by

$$\lambda_2 (a (\tau)) = \lambda a (\tau)^\beta$$

where $\lambda > 0$ is a constant, $a (\tau)$ is the social contact rate (i.e. the number of social contacts per unit of time) and $\beta \in (0, 1)$ is another constant. This specification of the arrival rate captures the idea that the risk of getting infected rises in the number of contacts the individual has. The more contacts, the higher the chance of getting infected. The recovery rate is assumed to be constant and denoted by λ_3.

Assume the three states are associated with corresponding health levels, h_1, h_2 and h_3. We assume for simplicity that healthy and recovered individuals enjoy the same health level. We also assume that the health level of a sick individual is lower. In short, $h_1 = h_3 > h_2$. Given these assumptions and letting $h (\tau)$ denote the health level of the individual under consideration at time t, we can model the transition between the health states by

$$dh (\tau) = (h_2 - h (\tau))\, dq_2 (\tau) + (h_3 - h (\tau))\, dq_3 (\tau) . \tag{4}$$

The objective function of the individual consists in an infinite horizon utility function $U (t)$ that denotes the present value of a stream of instantaneous utility $u (a (\tau))$. Just as the arrival rate for getting sick, utility also rises in contacts. This captures the idea that an individual is a social being that likes meeting people outside of the own household. A standard specification would read

$$u (a) = a^\sigma$$

with $0 < \sigma < 1$. Equipping the individual with a standard time-preference rate $\rho > 0$ yields the objective function

$$U (t) = \int_t^\infty e^{-\rho[\tau - t]} u (a (\tau))\, d\tau. \tag{5}$$

As the individual's state variable is health $h(\tau)$, the value of behaving optimally is given by $V(h(\tau)) = \max_{a(\tau)} U(\tau)$ subject to (8). After the usual steps, we obtain the Bellman equation

$$\rho V(h(\tau)) = \max_{a(\tau)} \{u(a(\tau)) + \lambda_2(a(\tau))[V(h_2) - V(h(\tau))] + \lambda_3[V(h_3) - V(h(\tau))]\}. \quad (6)$$

The first order condition when being healthy in state 1 reads

$$u'(a) = \lambda_2'(a)[V(h_2) - V(h_1)]. \quad (7)$$

In state 2 and 3, the number of social contacts are exogenous and denoted by a_2 and a_3.

1. Discuss the similarities and differences between the SIR model presented above and a typical search and matching model of unemployment.
2. Write down the health constraint of the individual for states 1, 2 and 3. Give an interpretation to them in words.
3. Consider the Bellman equation (10)
   (a) Describe it in words.
   (b) Derive the Bellman equation starting from
   $$\rho V(h(\tau)) = \max_{a(\tau)} \left\{u(a(\tau)) + \frac{1}{d\tau}E_t dV(h(\tau))\right\}.$$
4. Compute the value of being recovered, i.e. $V(h_3)$ by employing the objective function (9). *Hint: In this model, it is assumed that once the a sick individual is recovered, he remains recovered (questionable for Covid-19 but true for this exam).*
5. Compute the value of being healthy, $V(h_1)$, from the Bellman equation using the functional forms of the utility function and the arrival rate λ_2. Assume that $V(h_2) \equiv V_2$ is constant.
6. Consider the first-order condition (11).
   (a) Express (11) for the functional forms of the utility function and the arrival rate λ_2 by considering $V(h_2) - V(h_1)$ to be independent of a.
   (b) Now imagine the risk of turning sick goes up, i.e. λ increases. Does the number of contacts an individual chooses go up or down? Explain why.
   (c) Imagine it turns out that being sick is not so bad after all (technically: imagine V_2 rises). Does an individual choose to have more or less contacts? Explain why.
   (d) Now take the dependence of $V(h_1)$ on contacts a into account. Express the first-order condition using functional forms. *Hint: One should obtain an expression, which shows that a is an implicit function of the model parameters and V_2.*
   (e) What does this version of the first-order condition tell you about changes in the contact rate a if, say, λ increases? Are effects unambiguous now?

Solution Key

Question 1 - Green growth: non-renewable resources

1. We set up the Hamiltonian,

$$\mathbb{H} = u(A(t)R(t)) + \lambda(t)[-R(t)]$$

with FOCs

$$\frac{\partial \mathbb{H}}{\partial R(t)} = 0 \Leftrightarrow u'(C(t))A(t) = \lambda(t)$$

$$\dot{\lambda}(t) = \rho\lambda(t) - \frac{\partial \mathbb{H}}{\partial S(t)} \Leftrightarrow \dot{\lambda}(t) = \rho\lambda(t)$$

Taking the time derivative of the first FOC

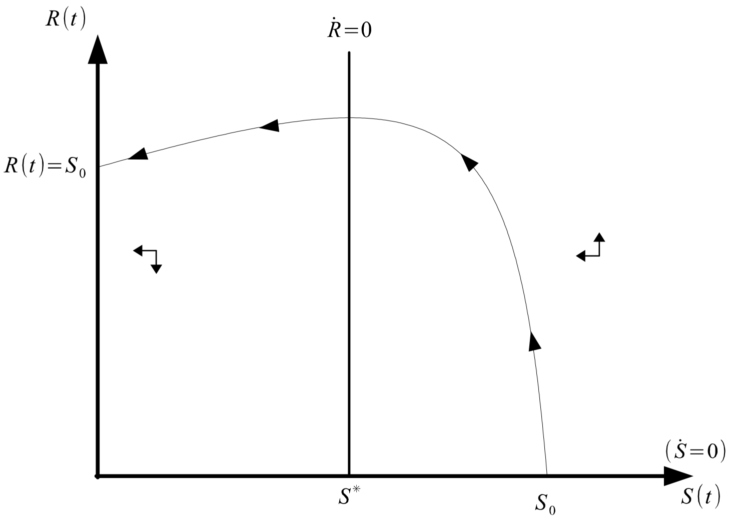
$$\begin{aligned}
\frac{d}{dt}\lambda(t) &= \frac{d}{dt}(u'(C(t))A(t)) \\
\dot{\lambda}(t) &= u''(C(t))A(t)[\dot{A}(t)R(t) + A(t)\dot{R}(t)] + u'(C(t))\dot{A}(t)
\end{aligned}$$

and now combining it with the second FOC, we obtain

$$\begin{aligned}
\dot{\lambda}(t) &= \lambda(t)\rho \\
u''(C(t))A(t)[\dot{A}(t)R(t) + A(t)\dot{R}(t)] + u'(C(t))\dot{A}(t) &= u'(C(t))A(t)\rho \\
u''^2\left[\frac{\dot{A}(t)}{A(t)}R(t) + \dot{R}(t)\right] + u'(C(t))\dot{A}(t) &= u'(C(t))A(t)\rho \\
u''(C(t))A(t)\left[gR(t) + \dot{R}(t)\right] + u'(C(t))\frac{\dot{A}(t)}{A(t)} &= u'(C(t))\rho \\
u''(C(t))A(t)\left[gR(t) + \dot{R}(t)\right] &= u'(C(t))[\rho - g] \\
-\frac{u''(C(t))}{u'(C(t))}A(t)\left[gR(t) + \dot{R}(t)\right] &= g - \rho
\end{aligned}$$

using the CRRA utility function we have

$$\begin{aligned}
-\frac{-\sigma C(t)^{-\sigma-1}}{C(t)^{-\sigma}}A(t)\left[gR(t) + \dot{R}(t)\right] &= g - \rho \\
\frac{\sigma}{C(t)}A(t)\left[gR(t) + \dot{R}(t)\right] &= g - \rho \\
\frac{\sigma}{A(t)R(t)}A(t)\left[gR(t) + \dot{R}(t)\right] &= g - \rho \\
\frac{\sigma}{R(t)}\left[gR(t) + \dot{R}(t)\right] &= g - \rho \\
\sigma\left[g + \frac{\dot{R}(t)}{R(t)}\right] &= g - \rho \\
\sigma g + \sigma\frac{\dot{R}(t)}{R(t)} &= g - \rho \\
\sigma\frac{\dot{R}(t)}{R(t)} &= (1-\sigma)g - \rho \\
\frac{\dot{R}(t)}{R(t)} &= \frac{(1-\sigma)g - \rho}{\sigma}
\end{aligned}$$



When individuals are sufficiently patient (i.e. ρ is sufficiently low), resource use will increase over time. When they are too impatient, they start at a high resource use that declines over time. Patient individuals shift resource use (and thereby consumption) into the future.

2. In order to draw the phase diagram in $R(t)$ and $S(t)$, we need to determine the zero-motion lines (ZMLs) of the system and the conditions for growth in each differential equation. The ZMLs are given by

$$\dot{R}(t) \geq 0 \Leftrightarrow \frac{(1-\sigma)g-\rho}{\sigma} \geq 0,$$
$$\dot{S}(t) \leq 0 \Leftrightarrow -R(t) \leq 0.$$

In turn, these imply for the ZML for $R(t)$ a straight vertical line, while the ZML for $S(t)$ does not exist, as we cannot have $-R(t) = 0$ (except in the long run). We can think of it as coinciding with the horizontal axis. Also, we note from before that when $S(t) = 0$ we have $R(t) = S_0$ and when $R(t) = 0$ we have $S(t) = S_0$.

3. Consumption growth is given by

$$\frac{\dot{C}(t)}{C(t)} = \frac{\dot{A}(t)}{A(t)} + \frac{\dot{R}(t)}{R(t)} = g + \frac{(1-\sigma)g-\rho}{\sigma} = \frac{g-\rho}{\sigma}.$$

Thus, consumption grows over time if the rate of technological innovation exceeds the time preference rate, i.e. $g > \rho$.

4. In order for this model to admit a solution where $\dot{C}(t) > 0$ and $\dot{R}(t) < 0$, we need to have to conditions be satisfied,

$$\frac{g-\rho}{\sigma} > 0 \text{ and } \frac{(1-\sigma)g-\rho}{\sigma} < 0.$$

For this to be true, we therefore need $g > \rho$ as well as

$$\frac{(1-\sigma)g-\rho}{\sigma} < 0 \Leftrightarrow (1-\sigma)g < \rho$$

This last condition is trivially satisfied if $\sigma > 1$, i.e. consumption is sufficiently inelastic over time and individuals find it difficult to shift consumption intertemporally. In general, it only requires that the time preference rate lie between $(1-\sigma)g < \rho < g$.

5. Starting from $R(t) = R_0 e^{g_R t}$ and inserting it into $\dot{S}(t) = -R(t)$, we obtain

$$\begin{aligned}\dot{S}(t) = -R_0 e^{g_R t} \Leftrightarrow S(t) &= S_0 - \int_0^t R_0 e^{g_R s} ds \\ &= S_0 - R_0 \int_0^t e^{g_R s} ds \\ &= S_0 - R_0 \left[\frac{e^{g_R s}}{g_R}\right]_0^t \\ &= S_0 - R_0 \frac{e^{g_R t} - 1}{g_R}\end{aligned}$$

Thus the stock at a point in time t will be given by the initial level minus the current level of resources use, which itself will depend on its original level R_0 and its growth rate between time 0 and time t.
When t goes to infinity, given that we have a negative growth rate of $R(t)$, the limit is given by

$$\begin{aligned}\lim_{t\to\infty} S(t) &= \lim_{t\to\infty} S_0 - R_0 \frac{e^{g_R t} - 1}{g_R} \\ &= S_0 - R_0 \frac{-1}{g_R} \\ &= S_0 + \frac{R_0}{g_R} < S_0\end{aligned}$$

where the last inequality holds because $g_R < 0$. Thus, in the long-run, the stock of resources will be reduced by R_0/g_R. If this term is sufficiently large, such that $\lim_{t\to\infty} S(t) < 0$, then the stock would be depleted completely in finite time. If however, the term is sufficiently small, then the economy would converge toward a strictly positive stock level.

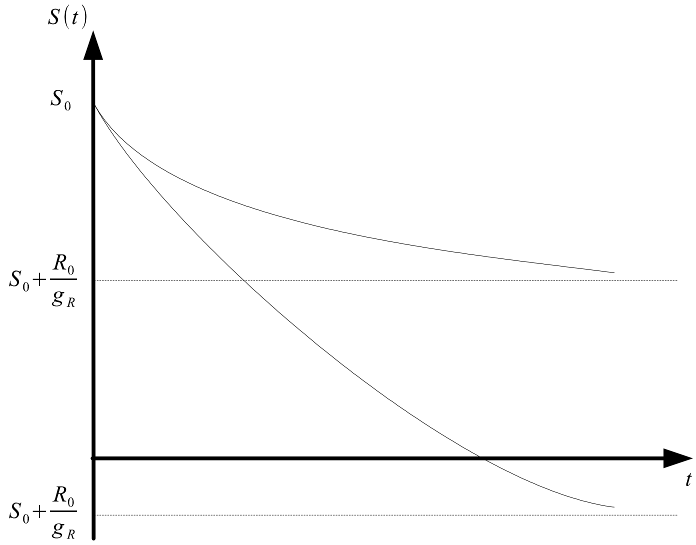
6. Plotting $S(t) = S_0 - R_0 \frac{e^{g_R t} - 1}{g_R}$ over time, we can see that the long-run value of $S(t)$ can remain positive if R_0 is small enough or $|g_R|$ is sufficiently large (upper solid line). The stock would be depleted in finite time if $|g_R|$ is sufficiently small or R_0 sufficiently large (lower solid line).

Now we can compute the point in time t when $S(t)$ will be completely used up. For this we start from the condition

$$S(t) = 0 \Leftrightarrow S_0 = R_0 \frac{e^{g_R t} - 1}{g_R}.$$

Now solving for t, we obtain

$$\begin{aligned}S_0 &= R_0 \frac{e^{g_R t} - 1}{g_R} \\ \Leftrightarrow S_0 g_R &= R_0 [e^{g_R t} - 1] \\ \Leftrightarrow \frac{S_0}{R_0} g_R + 1 &= e^{g_R t} \\ \Leftrightarrow g_R t &= \ln\left[\frac{S_0}{R_0} g_R + 1\right] \\ \Leftrightarrow t &= \frac{1}{g_R} \ln\left[\frac{S_0}{R_0} g_R + 1\right]\end{aligned}$$



This result holds true when $g_R < 0$ as the term inside of the log operator would be strictly smaller than 1, thus implying the log to be negative also, and giving us the product of two negative terms.
Given that S_0 and R_0 are initial conditions, the only term that can be affected is g_R. In order to increase t, we would need to increase g_R in absolute terms, i.e. increase the rate at which resources use decrease over time, such that we may converge to a positive $S(t)$ as seen in the figure above.

Question 2 - Epidemics and decisions

Consider an individual who can be in three states: healthy, sick and recovered. In more traditional epidemiological terms, the states would be called “suzeptible”, “infectious” and “removed” - which gives the acronym SIR. For simplicity, we denote the states by 1, 2 and 3, respectively. The individual is initially in state 1, at some point moves into state 2 and then ends up in state 3, as illustrated in Figure 2. Technically, state 1 is an emitting state (no individual ever enters this state) and state 3 is an absorbing state (no individual ever leaves this state).

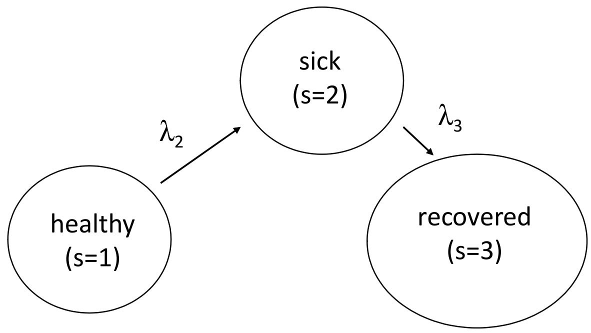


Figure 2 *The SIR model*

The rate for the individual to get infected is given by

$$\lambda_2 \left(a\left(\tau\right)\right) = \lambda a\left(\tau\right)^{\beta}$$

where $\lambda > 0$ is a constant, $a(\tau)$ is the social contact rate (i.e. the number of social contacts per unit of time) and $\beta \in (0,1)$ is another constant. This specification of the arrival rate captures the idea that the risk of getting infected rises in the number of contacts the individual has. The more contacts, the higher the chance of getting infected. The recovery rate is assumed to be constant and denoted by λ_3.

Assume the three states are associated with corresponding health levels, h_1, h_2 and h_3. We assume for simplicity that healthy and recovered individuals enjoy the same health level. We also assume that the health level of a sick individual is lower. In short, $h_1 = h_3 > h_2$. Given these assumptions and letting $h(\tau)$ denote the health level of the individual under consideration at time t, we can model the transition between the health states by

$$dh(\tau) = (h_2 - h(\tau))\, dq_2(\tau) + (h_3 - h(\tau))\, dq_3(\tau). \tag{8}$$

The objective function of the individual consists in an infinite horizon utility function $U(t)$ that denotes the present value of a stream of instantaneous utility $u(a(\tau))$. Just as the arrival rate for getting sick, utility also rises in contacts. This captures the idea that an individual is a social being that likes meeting people outside of the own household. A standard specification would read

$$u(a) = a^{\sigma}$$

with $0 < \sigma < 1$. Equipping the individual with a standard time-preference rate $\rho > 0$ yields the objective function

$$U(t) = \int_t^{\infty} e^{-\rho[\tau - t]} u(a(\tau))\, d\tau. \tag{9}$$

As the individual's state variable is health $h(\tau)$, the value of behaving optimally is given by $V(h(\tau)) = \max_{a(\tau)} U(\tau)$ subject to (8). After the usual steps, we obtain the Bellman equation

$$\rho V(h(\tau)) = \max_{a(\tau)} \{u(a(\tau)) + \lambda_2(a(\tau))[V(h_2) - V(h(\tau))] + \lambda_3[V(h_3) - V(h(\tau))]\}. \tag{10}$$

The first order condition when being healthy in state 1 reads

$$u'(a) = \lambda_2'(a)[V(h_1) - V(h_2)]. \tag{11}$$

In state 2 and 3, the number of social contacts are exogenous and denoted by a_2 and a_3.

1. Discuss the similarities and differences between the SIR model presented above and a typical search and matching model of unemployment.
   Answer: Similarity: individuals in both models face risks governed by Poisson processes. The differences are as follows

SIR	search and matching
3 states	2 states
an emitting state and an absorbing state	unemployment risk
...	...

2. Write down the health constraint of the individual for states 1, 2 and 3. Give an interpretation to them in words.
   Answer:

   state 1: $dh(\tau) = (h_2 - h_1)\, dq_2(\tau)$.
   The health level of a healthy individual
   will decrease by $|h_2 - h_1|$ when he is infected

state 2: $dh(\tau) = (h_3 - h_2)\, dq_3(\tau)$.
The health level of a sick individual
will increase by $h_3 - h_2$ when he recovers

state 3: $dh(\tau) = 0$.
The health level of a recovered individual
remains constant since he never gets infected again

3. Consider the Bellman equation (10)

   (a) Describe it in words.
   Answer: the flow value of the optimal contact rate is equal to instantaneous utility minus the expected utility loss due to infection plus the expected utility gain due to recovery.

   (b) Derive the Bellman equation starting from

$$\rho V(h(\tau)) = \max_{a(\tau)} \left\{ u(a(\tau)) + \frac{1}{d\tau} E_t dV(h(\tau)) \right\}.$$

   Answer:

$$\begin{aligned} dV(h(\tau)) &= [V(h_2) - V(h(\tau))]\, dq_2(\tau) + [V(h_3) - V(h(\tau))]\, dq_2(\tau) \\ E_t dV(h(\tau)) &= \lambda_2(a)\,[V(h_2) - V(h(\tau))]\, d\tau + \lambda_3\,[V(h_3) - V(h(\tau))]\, d\tau \end{aligned}$$

   Thus,

$$\begin{aligned} \rho V(h(\tau)) &= \max_{a(\tau)} \left\{ u(a(\tau)) + \frac{\lambda_2(a(\tau))\,[V(h_2) - V(h(\tau))]\, d\tau + \lambda_3\,[V(h_3) - V(h(\tau))]\, d\tau}{d\tau} \right\} \\ \rho V(h(\tau)) &= \max_{a(\tau)} \{ u(a(\tau)) + \lambda_2(a(\tau))\,[V(h_2) - V(h(\tau))] + \lambda_3\,[V(h_3) - V(h(\tau))] \} \end{aligned}$$

4. Compute the value of being recovered, i.e. $V(h_3)$ by employing the objective function (9). *Hint: In this model, it is assumed that once the a sick individual is recovered, he remains recovered (questionable for Covid-19 but true for this exam).*
   Answer:

$$\begin{aligned} V(h_3) &= \int_t^\infty e^{-\rho[\tau - t]} u(a_3)\, d\tau \\ &= \int_t^\infty e^{-\rho[\tau - t]} a_3^\sigma d\tau = a_3^\sigma \int_t^\infty e^{-\rho[\tau - t]} d\tau = a_3^\sigma \frac{1}{-\rho} e^{-\rho[\tau - t]}|_t^\infty \\ &= a_3^\sigma \frac{1}{-\rho}(0 - 1) = \frac{a_3^\sigma}{\rho} \end{aligned}$$

5. Compute the value of being healthy, $V(h_1)$, from the Bellman equation using the functional forms of the utility function and the arrival rate λ_2. Assume that $V(h_2) \equiv V_2$ is constant.
   Answer: The Bellman equation in state 1 is given by

$$\begin{aligned} \rho V(h_1) &= u(a) + \lambda_2(a)\,[V_2 - V(h_1)] \\ V(h_1) &= \frac{u(a) + \lambda_2(a)\, V_2}{\rho + \lambda_2(a)}. \\ V(h_1) &= \frac{a^\sigma + \lambda a^\beta V_2}{\rho + \lambda a^\beta} \end{aligned}$$

6. Consider the first-order condition (11).

   (a) Express (11) for the functional forms of the utility function and the arrival rate λ_2 by considering $V(h_2) - V(h_1)$ to be independent of a.
   Answer: The FOC reads

$$\sigma a^{\sigma-1} = \lambda \beta a^{\beta-1} [V(h_1) - V_2] \Leftrightarrow$$
$$a^{\sigma-\beta} = \lambda \frac{\beta}{\sigma} [V(h_1) - V_2].$$

   (b) Now imagine the risk of turning sick goes up, i.e. λ increases. Does the number of contacts an individual chooses go up or down? Explain why.
   Answer: a rises if $\sigma > \beta$ (the contact elasticity of utility is greater than that of the infection rate) and falls if $\sigma < \beta$ (the contact elasticity of utility is less than that of the infection rate).

   (c) Imagine it turns out that being sick is not so bad after all (technically: imagine V_2 rises). Does an individual choose to have more or less contacts? Explain why.
   Answer: a rises if $\sigma < \beta$ (the contact elasticity of utility is less than that of the infection rate) and falls if $\sigma > \beta$ (the contact elasticity of utility is bigger than that of the infection rate).

   (d) Now take the dependence of $V(h_1)$ on contacts a into account. Express the first-order condition using functional forms. *Hint: One should obtain an expression, which shows that a is an implicit function of the model parameters and V_2.*
   Answer: The FOC reads

$$a^{\sigma-\beta} = \lambda \frac{\beta}{\sigma} [V(h_1) - V_2] = \lambda \frac{\beta}{\sigma} \left[\frac{a^\sigma + \lambda a^\beta V_2}{\rho + \lambda a^\beta} - V_2 \right] \Leftrightarrow$$
$$\left(\rho + \lambda a^\beta\right) a^{\sigma-\beta} = \lambda \frac{\beta}{\sigma} \left[a^\sigma + \lambda a^\beta V_2 - \left(\rho + \lambda a^\beta\right) V_2\right] \Leftrightarrow$$
$$\rho a^{\sigma-\beta} + \lambda a^\sigma = \lambda \frac{\beta}{\sigma} [a^\sigma - \rho V_2] \Leftrightarrow$$
$$a^{\sigma-\beta} = \left(\frac{\beta}{\sigma} - 1\right) \frac{\lambda}{\rho} a^\sigma - \lambda \frac{\beta}{\sigma} V_2$$

$$\sigma a^{\sigma-1} = \lambda \beta a^{\beta-1} [V(h_1) - V_2]$$
$$\sigma a^{\sigma-1} = \lambda \beta a^{\beta-1} \left[\frac{a^\sigma + \lambda a^\beta V_2}{\rho + \lambda a^\beta} - V_2 \right]$$
$$1 = \frac{\lambda \beta}{\sigma} a^{\beta-\sigma} \left[\frac{a^\sigma + \lambda a^\beta V_2}{\rho + \lambda a^\beta} - V_2 \right]$$

   (e) What does this version of the first-order condition tell you about changes in the contact rate a if, say, λ increases? Are effects unambiguous now?
   Answer: The expression shows that a is an implicit function of λ and other parameters. To understand the effect of λ on a, one would need to compute $\frac{\partial a}{\partial \lambda}$. When we denote

$$G(\bullet) = 1 - \frac{\lambda \beta}{\sigma} a^{\beta-\sigma} V_2 \left[\frac{a^\sigma + \lambda a^\beta}{\rho + \lambda a^\beta} - 1 \right]$$

   The effect of λ on a can be measured by

$$\frac{\partial a}{\partial \lambda} = -\frac{\partial G / \partial \lambda}{\partial G / \partial a}.$$

   The effects are ambiguous since the derivative depends on, among others, $\beta - \sigma$.