Johannes Gutenberg-University Mainz
Bachelor of Science in Wirtschaftswissenschaften

Wealth distributions - Problem Set 2

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Niklas Scheuer

www.macro.economics.uni-mainz.de

Exercise 1 - Properties of the CRRA utility function

a) What is intertemporal elasticity of substitution of the following utility function

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma}$$

and how is it determine?

b) Logarithmic utility function – What is the limit of $\frac{c^{1-\sigma}-1}{1-\sigma}$ as σ tends to 1? Use l'Hôpital's rule plot the function qualitatively for increasing levels of σ.

1. Reminder: l'Hôpital's rule says that if $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ and $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$.

Exercise 2 - Closed-form solution for a CRRA utility function

Let households maximise a CRRA-utility function

$$U_t = E_t \left[\gamma c_t^{1-\sigma} + (1-\gamma) c_{t+1}^{1-\sigma} \right] = \gamma c_t^{1-\sigma} + (1-\gamma) E_t c_{t+1}^{1-\sigma}, \tag{1}$$

where the interest rate in period $t+1$ and, hence, consumption in $t+1$ is uncertain. This is maximised, subject to budget constraints, i.e.

$$w_t = c_t + s_t; (1 + r_{t+1}) s_t = c_{t+1} \tag{2}$$

Show that an optimal consumption-saving decision given budget constraints (2) implies savings of

$$s_t = \frac{w_t}{1 + \left\{ \frac{\gamma}{(1-\gamma)\phi} \right\}^{\varepsilon}}, \tag{3}$$

where $\varepsilon = 1/\sigma$ is the intertemporal elasticity of substitution and $\phi \equiv E_t \left[(1 + r_{t+1})^{1-\sigma} \right]$ is the expected (transformed) interest rate. Show further that consumption when old is

$$c_{t+1} = (1 + r_{t+1}) \frac{w_t}{1 + \left\{ \frac{\gamma}{(1-\gamma)\phi} \right\}^{\varepsilon}} \tag{4}$$

and that consumption of the young is

$$c_t = \frac{\left\{ \frac{\gamma}{(1-\gamma)\phi} \right\}^{\varepsilon} w_t}{1 + \left\{ \frac{\gamma}{(1-\gamma)\phi} \right\}^{\varepsilon}} \tag{5}$$

Exercise 4 - Another basic OLG model

Turning again to a OLG model, a household born in t maximises lifetime utility

$$\max_{c_t^y, c_{t+1}^o} \ln c_t^y + \beta \ln c_{t+1}^o,$$

with $0 < \beta < 1$, subject to the following constraints in period t and $t+1$,

$$w_t - s_t = c_t^y; r_{t+1} s_t = c_{t+1}^o$$

We must also not the results from the maximisation problem of the production sector, giving us optimal factor demand in equilibrium,

$$\begin{aligned} w_t &= f(k_t) - k_t f'(k_t), \\ r_{t+1} &= f'(k_t) \end{aligned}$$

with $f(k_t) = k_t^{\alpha}$, and $0 < \alpha < 1$, and population is constant in size.

a) Derive the Euler equation in the basic OLG model.

b) What is the optimal level of savings needed to safisfy the Euler equation from above?

c) Using the market-clearing condition $k_{t+1} = s_t$, what is the capital accumulation function giving k_{t+1} as a function of k_t?