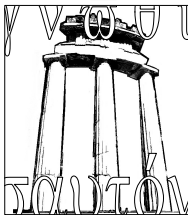


Klaus Wälde

# Applied Intertemporal Optimization

Edition 1.2 plus: Textbook and Solutions Manual



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I teach

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we develop a 2-period OLG model with stochastic death in class